*All candidates should attempt **3** questions. If more than 3 questions are attempted only the best 3 will be counted.*

Only resit candidates may attempt questions 6 and 7.

1. Write down the truth tables for the Boolean operators ¬(not), • (and), and + (or).

[6 marks]

Using these truth tables derive De Morgan's laws

$$\neg(a + b) = (\neg a)\bullet(\neg b)$$

$$\neg(a\bullet b) = (\neg a) + (\neg b)$$

[4 marks]

What do you understand by 'conjunctive form' and 'disjunctive form' in relation to a Boolean expression? Outline a method that allows conversion between such forms. Why are conjunctive and disjunctive forms useful?

[11 marks]

Show that a Boolean expression can be rewritten so as to use *either* the operators ¬ and • only *or* the operators ¬ and + only.

[12 marks]

2 Explain carefully what is meant by *induction* and *strong induction*.

[9 marks]

A binary tree is defined as a collection of *nodes* and *branches*, which link these nodes, so that every node apart from one, called the *root*, has an incoming branch from another node and 0 or 2 outgoing branches to other nodes. (*Incoming* and *outgoing* are interpreted as 'nearer the root' and 'going further away from the root'.) Nodes with 2 outgoing nodes are called *internal nodes*, the others are *external nodes*. There are n internal nodes and s external nodes.

Prove *by induction on n* that $s = n + 1$

[6 marks]

The number of nodes, excluding the root, on the path from the root to another node is called the *path length* of that node. The *internal path length* of the tree, *I(n),* is the

sum of all path lengths to internal nodes. The *external path length, E(n)*, is the sum of all path lengths from the root to the external nodes.

Prove *by induction* that $E(n) = I(n) + 2n$

[6 marks]

By regarding the tree as a root with two (sub)trees attached *prove* these 2 results using *strong induction.*

[12 marks]

3 Peano's axioms for arithmetic with natural numbers are

- 0 is a natural number;
- $\forall x.x \rightarrow \exists x'$;
- $\forall x \forall y$. $x' = y' \vdash x = y$;
- $\neg \exists x$. $x' = 0$;
- $p(0) \wedge (\forall x.p(x) \rightarrow p(x')) \vdash \forall x.p(x)$;

Explain briefly what these axioms mean.

[5 marks]

Addition and multiplication are defined by

Addition: $x + 0 = x$; $x + y' = (x + y)'$

Multiplication: $x \bullet 0 = 0$; $x \bullet y' = x \bullet y + x$

Prove that multiplication is commutative, $x \bullet y = y \bullet x$

You *must* prove all intermediate results you need and should annotate all proofs as required. (You may assume that addition is commutative and associative.)

[28 marks]

4. What do you understand by the *pigeon hole principle*?

[5 marks]

Prove that all the rational numbers, $^{p}/_{q}$, where p and q are positive integers, with $q \neq 0$, have a decimal expansion that either terminates or is ultimately periodic.

[13 marks]

Secondly, *prove* that *only* rational numbers have this property.

[13 marks]

What can you therefore conclude about the irrational number $\sqrt{2}$?

[2 marks]

5 Define *countability* of an infinite set. Give examples of 2 countable sets (other than those in the next part of the question) and show that they are countable.

[13 marks]

A *word* is defined as a sequence of letters chosen from a countable infinite set of letters. Show that the following sets of words are countable:

i) two letter words;

ii) three letter words;

iii) *n* letter words for fixed *n*;

iv) all words of finite length.

[20 marks]

Only resit candidates may attempt the next 2 questions

6 What is the difference between the *syntax* and the *semantics* of a formal system (such as propositional logic)? Are valid conclusions always true? If not, what is the difference between validity and truth in logic?

[6 marks]

How is the truth of a sentence confirmed in propositional logic? Demonstrate by showing that the sentence $P \wedge (\neg Q \vee R)$ is a contingent truth.

[5 marks]

Funny logic is defined as follows:

The language consists of the symbol * and the connectives % and & .

The grammar consists of the following two definitions:

sentence ::= string of stars % string of stars & string of stars

string of stars ::= * | string of stars *

The deductive system consists of the two inference rules (where a, b, c are strings of stars):

a % b & c

a % b* & c*

and the sentence * % * & ** is a premise.

Propose an interpretation of the strings of stars and the connectives, and explain the inference rules and premise in the light of your interpretation.

[10 marks]

Does the sentence *** % **** & ******* make sense in your interpretation? Can it be derived from the deductive system? Give your reasons?

[10 marks]

Construct a proof of the derivation *** % ** & ***** ├ *** % **** & *******

[3 marks]

Propose a new rule which will allow *** % **** & ******* to be derived. How is the rule to be interpreted? Use it to prove * % * & ** ├ *** % **** & ******* .

[6 marks]

7 Explain the inference rules known as *modus ponens* and *modus tollens*.

[6 marks]

Consider the following premises about configurations of boxes on a table:

$\forall x \forall y \forall z$ ((above(x, y) $\wedge$ above(y, z) $\rightarrow$ above(x, z))

$\forall x \forall y$ (on(x ,y) $\rightarrow$ above(x, y))

on(*a, table*)

on (*c, table*)

on(*b, a*)

What do these statements mean?

[3 marks]

Construct a proof that *b* is above the *table*. All inference rules should be labelled and numbered.

[5 marks]

Add further premises to express the fact that two boxes are at the *same level* if they are on the same object and *prove* that *a* and *c* are at the same level.

[5 marks]

Add further premises that 2 boxes are at the same level if the things they are on are at the same level. Then add two further boxes, *d* and *e*, with *d* on the *table* and *e* on *d*. *Prove* that *b* and *e* are at the same level.

[7 marks]

If *x* is on *y* then the level of *x* is 1 more than the level of *y*. The *table* is at level 0. Express these level rules in symbolic terms. Give the levels of *a, b, c, d* and *e*.

[7 marks]

END OF PAPER