2.

a) Give a mathematical definition of the *order notation*

$$f(n) \in O(g(n))$$

and explain how this concept relates to the algorithmic idea of *worst case analysis*.

[6 marks]

b) Are the following statements true or false? Justify your answers using a careful argument based on the mathematical definition of ’O’ notation. (You may assume that $n > 0$.)

(i) $2^{n+1} \in O(2^n)$

(ii) $2^{2n} \in O(2^n)$

(iii) $\log(a^n) \in O(n)$

[9 marks]

c) Use a simple graphical argument to show that the discrete sum

$$\sum_{i=1}^{n} f(i)$$

is bounded above by the integral

$$\int_{1}^{n+1} f(t)\, dt$$

provided that $f(t)$ is a non-decreasing function.

[5 marks]

d) Solve the following recurrence relations, simplifying your final answers using ’O’ notation (you may assume that n is a power of 2 where appropriate):

(i)
$$\begin{aligned} f(0) &= 0 \\ f(1) &= 2 \\ f(n) &= 4f(n-1) - 3f(n-2), \quad n > 1 \end{aligned}$$

[4 marks]

(ii)
$$\begin{aligned} f(0) &= 1 \\ f(1) &= 4 \\ f(n) &= 4f(n-1) - 4f(n-2), \quad n > 1 \end{aligned}$$

[4 marks]

(iii)
$$\begin{aligned} f(1) &= 0 \\ f(n) &= 4f(\frac{n}{2}) + n, \quad n > 1 \end{aligned}$$

[5 marks]

[Total 33 marks]

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