UNIVERSITY OF SURREY©

M. Math. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MS320 Function Spaces

Time allowed – 2 hrs

Autumn Semester 2007

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

(a) State the Contraction Theorem. [3]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction in the usual metric $d(x, y) = |x - y|$.
Prove that f is uniformly continuous. [4]

(c) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$F(x, y) = \left(\frac{1}{3}(x + \cos y)\ ,\ \frac{1}{3}(y + \cos x) \right).$$

Show that F has a unique fixed point. [5]

(d) Let $A = (a_{i,j})_{i,j=1}^n$ be a real $n \times n$-matrix such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{i,j}|^2 < 1. \tag{1}$$

Show that the linear equation

$$Ax + b = x$$

has a unique solution in $\mathbb{R}^n$. (Hint: show that an appropriate map is a contraction w.r.t. Euclidean metric.) [6]

(e) Conclude that for any matrix satisfying (1)

$$\det(I - A) \neq 0,$$

where I is the $n \times n$ identity matrix. [3]

(f) Suppose $F : X \to X$ is a mapping of a complete metric space (X, d), and assume that for some $n > 1$, the n-fold composition

$$F^n := \underbrace{F \circ \cdots \circ F}_{n \text{ times}}$$

is a contraction. Does this imply that F has a fixed point? Is it unique?
Justify your answer. [4]

Question 2

(a) What is an orthonormal basis of a Hilbert space? [3]

(b) Let f belong to Hilbert space $(X, \langle\ ,\ \rangle)$ with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$.
Show that the Fourier coefficients of f tend to zero. [4]

Let $f \in L^2([0,1])$ be given by $f(x) = \sin \pi x$. The Fourier series of f is given by

$$F(x) = a_0 + \sum_{k\in\mathbb{N}} a_k \cos 2\pi kx + \sum_{k\in\mathbb{N}} b_k \sin 2\pi kx.$$

(c) Give the standard orthonormal basis for $L^2([0,1])$ and use it to:
(i) compute a_0,
(ii) explain (without explicitly computing integrals) why $b_k = 0$ for all $k \in \mathbb{N}$. [5]

(d) Use integration by parts to show that $a_k = \frac{4}{\pi(1-4k^2)}$ for $k \geq 1$. [4]

(e) State Dirichlet's theorem. [4]

(f) Using the above, compute

$$\sum_{k\in\mathbb{N}} \frac{1}{4k^2-1}.$$

Explain the role of Dirichlet's theorem in your answer. [5]

Question 3

Let $(\ell^2, \langle\ ,\ \rangle)$ be the standard Hilbert space of sequences.

(a) The system $\{d^n\}_{n\in\mathbb{N}}$ in ℓ^2 is given by

$$d^1 = b\left(a, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right)$$

$$d^2 = b\left(0, a, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right)$$

$$d^3 = b\left(0, 0, a, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right)$$

and in general

$$d^n_k = \begin{cases} 0 & \text{if } k < n; \\ ba & \text{if } k = n; \\ b2^{-(k-n-1)} & \text{if } k > n. \end{cases}$$

Find $a, b \in \mathbb{R}$ such that $\{d^n\}_{n\in\mathbb{N}}$ is orthonormal. [4]

(b) Give two equivalent statements to: $\{e_n\}_{n\in\mathbb{N}}$ is a complete system. [4]

(c) Is $\{d^n\}_{n\in\mathbb{N}}$ complete? Explain your answer.
(Hint: try $x = (1, t, t^2, t^3, \dots)$ for a well-chosen $t \in \mathbb{R}$.) [4]

(d) What is the definition of a separable space? [2]

(e) Let

$$X_n = \left\{x \in \ell^2 : x_k \in \mathbb{Q} \text{ for } k \le n \text{ and } x_k = 0 \text{ for } k > n\right\}.$$

Show that X_n is countable.
Hence show that ℓ^2 is separable. [6]

(f) Show that

$$Y = \{\{x_n\}_{n\in\mathbb{N}} : x_n = 0 \text{ or } 1\}$$

is uncountable.
It seems that the balls

$$B(y; \frac{1}{3}) = \{z : \|y - z\|_2 < \frac{1}{3}\}$$

for $y \in Y$ are all disjoint, so a dense subset D in ℓ^2 must have at least one element in each $B(y; \frac{1}{3})$. Why does this **not** contradict that ℓ^2 is separable? [5]

Question 4

(a) Let (X, d) be a metric space. Define what it means that $A \subset X$ is a compact set. [2]

(b) Prove that every compact set is bounded. [4]

(c) When is a metric space complete?
The metric d_* on $\mathbb{R}$ is defined as

$$d(x, y) = \arctan(|x - y|).$$

Is $(\mathbb{R}, d_*)$ a complete metric space? Explain your answer. [5]

(d) State the Heine-Borel Theorem for sets in Euclidean space $\mathbb{R}^n$.

Show that in $\mathbb{R}$, the Heine-Borel Theorem ceases to hold if the Euclidean metric is replaced by d_*. [5]

(e) Show that the closed unit ball in an infinite dimensional Hilbert space is not compact. [4]

(f) Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1-function such that

$$f'(x) \to 0 \text{ as } x \to \pm\infty.$$

Show that f is uniformly continuous. [5]

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