UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MS310 Introduction to Function Spaces

Time allowed – 2 hrs Autumn Semester 2007

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

(a) State the Contraction Theorem. [3]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction in the usual metric $d(x, y) = |x - y|$.
Prove that f is uniformly continuous. [4]

(c) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$F(x, y) = \left(\frac{1}{3}(x + \cos y) \; , \; \frac{1}{3}(y + \cos x) \right).$$

Show that F has a unique fixed point. [5]

(d) Let $A = (a_{i,j})_{i,j=1}^{n}$ be a real $n \times n$-matrix such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{i,j}|^2 < 1. \qquad (1)$$

Show that the linear equation

$$Ax + b = x$$

has a unique solution in $\mathbb{R}^n$. (Hint: show that an appropriate map is a contraction w.r.t. Euclidean metric.) [6]

(e) Conclude that for any matrix satisfying (1)

$$\det(I - A) \neq 0,$$

where I is the $n \times n$ identity matrix. [3]

(f) Suppose $F : X \to X$ is a mapping of a complete metric space (X, d), and assume that for some $n > 1$, the n-fold composition

$$F^n := \underbrace{F \circ \cdots \circ F}_{n \text{ times}}$$

is a contraction. Does this imply that F has a fixed point? Is it unique?
Justify your answer. [4]

Question 2

(a) Give the definition of $L^2([-1,1])$, and indicate for which $p \in \mathbb{R}$ the function

$$f(x) = \begin{cases} |x|^{-p} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

belongs to $L^2([0,1])$. [3]

(b) Show that if $f_n \to f$ uniformly, then $f_n \to f$ in $\| \ \|_2$. [3]

(c) Let $\{g_n\}_{n\in\mathbb{N}} \subset L^2([-1,1])$ be given by

$$g_n(x) = \begin{cases} n(1 - n|x|) & \text{if } |x| \leq \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Determine whether $\{g_n\}_{n\in\mathbb{N}}$ converges
(i) pointwise,
(ii) uniformly,
(iii) in $\| \ \|_2$. [6]

(d) Define what is meant by a Hilbert space, and explain why $(L^2([-1,1]), \| \ \|_2)$ is a Hilbert space, but $(L^1([-1,1]), \| \ \|_1)$ is not. [5]

(e) State the Cauchy-Schwarz inequality and explain what role it plays in showing that $L^2([-1,1])$ is a normed space. [4]

(f) Let $\langle \ , \ \rangle$ be the standard inner product on $L^2([-1,1])$.

Show that if $f_n \to f$ in $\| \ \|_2$, then

$$\langle f_n, h\rangle \to \langle f, h\rangle$$

for every $h \in L^2([-1,1])$. [4]

Question 3

(a) What is a Lipschitz continuous function between metric spaces? [2]

(b) Show that

$$\|f\|_* = \sup\left\{\frac{|f(x)-f(y)|}{|x-y|} : x,y \in [-\pi,\pi], x \neq y\right\}$$

is not a norm on the space $C([-\pi,\pi])$ of continuous functions. Give two different reasons. [4]

(c) Let X be the space of real continuously differentiable 2π-periodic functions such that $f(0)=0$. You may assume that $\| \ \|_*$ is a norm on X. Show that every function in X is bounded. [4]

(d) Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X, \| \ \|_*)$, and let f be the pointwise limit of $\{f_n\}_{n\in\mathbb{N}}$. Show that f is Lipschitz.

Is $(X, \| \ \|_*)$ a Banach space? Justify your answer. [6]

(e) What is meant by equivalent norms?

Show that $\| \ \|_1$ and $\| \ \|_\infty$ are equivalent on $\mathbb{R}^2$ [4]

(f) Show that $\| \ \|_2$ and $\| \ \|_*$ are not equivalent on the space X from item (c). [5]

Question 4

(a) What is an orthonormal basis of a Hilbert space? [3]

(b) Let f belong to Hilbert space $(X, \langle\ ,\ \rangle)$ with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$.
Show that the Fourier coefficients of f tend to zero. [4]

Let $f \in L^2([0,1])$ be given by $f(x) = \sin \pi x$. The Fourier series of f is given by

$$F(x) = a_0 + \sum_{k\in\mathbb{N}} a_k \cos 2\pi kx + \sum_{k\in\mathbb{N}} b_k \sin 2\pi kx.$$

(c) Give the standard orthonormal basis for $L^2([0,1])$ and use it to:
(i) compute a_0,
(ii) explain (without explicitly computing integrals) why $b_k = 0$ for all $k \in \mathbb{N}$. [5]

(d) Use integration by parts to show that $a_k = \frac{4}{\pi(1-4k^2)}$ for $k \geq 1$. [4]

(e) State Dirichlet's theorem. [4]

(f) Using the above, compute

$$\sum_{k\in\mathbb{N}} \frac{1}{4k^2-1}.$$

Explain the role of Dirichlet's theorem in your answer. [5]

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