Geodesics and orbits

Geodesics

We have seen that, when expressed in terms of an affine parameter s (which, for a timelike curve could be the proper time), the equation for geodesics can be written

$$\frac{d^2 x^a}{ds^2} + \Gamma^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = 0$$

where

$$g_{ab} \frac{dx^a}{ds} \frac{dx^b}{ds} = \begin{cases} +1 & \text{timelike} \\ 0 & \text{null} \\ -1 & \text{spacelike} \end{cases}$$

Given a set of coordinates and the spacetime metric we can solve these equations to determine the motion of test particles and light rays.

There is, however, an alternative approach to this problem.

Calculus of variations

Many physical laws can be derived from a "variational" approach. The basic idea is that a system tends to a state of minimum energy. By finding a suitable expression for this energy and minimising it, one can obtain the equations of motion.

Variational calculus was first developed for classical mechanics. In that case, one considers an action integral

$$I = \int_{t=p}^{q} L(x(t), \dot{x}(t))dt \qquad \text{with} \quad x(p) = k_1 \quad \text{and} \quad x(q) = k_2$$

Varying the "path" one can show that the action is minimised if the Euler-Lagrange equations are satisfied. These equations can be written;

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

When L depends on a curve $x^a(t)$ this generalises to;

$$I = \int_{t=p}^{q} L(x^a(t), \dot{x}^a(t))dt \qquad \Rightarrow \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^a} - \frac{\partial L}{\partial x^a} = 0$$

Geodesics (again)

We will use the variational approach to determine geodesics.

Since we want the geodesics to represent the "shortest path" in spacetime, we wish to minimise

$$I = \int ds \qquad \Rightarrow \qquad \frac{ds}{dt} = \left[g_{bc}\dot{x}^b \dot{x}^c \right]^{1/2} = L\left(x^a(t), \dot{x}^a(t) \right)$$

One can show that this leads us back to the geodesic equation.

It is, however, more practical (since one avoids the square-root) to work with

$$I = \int ds^2 = \tfrac{1}{2}\int g_{bc}\dot{x}^b \dot{x}^c ds = \int L ds$$

Then we have

$$\frac{\partial L}{\partial \dot{x}^a} = g_{ab}\dot{x}^b \qquad \Rightarrow \qquad \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^a} \right) = \partial_c g_{ab}\dot{x}^b \dot{x}^c + g_{ab}\ddot{x}^b$$

$$\frac{\partial L}{\partial x^a} = \partial_a g_{bc}\dot{x}^b \dot{x}^c$$

And the Euler-Lagrange equations lead to the same equations of motion as before.

Schwarzschild orbits

Let us apply this method to the problem of determining the geodesics of the Schwarzschild spacetime. Then we have

$$g_{ab}dx^a dx^b = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

To find the timelike geodesics, we take

$$2L = g_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} = g_{ab}\dot{x}^a\dot{x}^b$$

which leads to

$$2L = g_{ab}\dot{x}^a\dot{x}^b = \left(1-\frac{2M}{r}\right)\dot{t}^2 - \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\ \dot{\phi}^2\right)$$

Use this together with the Euler-Lagrange equations

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^a} - \frac{\partial L}{\partial x^a} = 0$$

Taking *a=0* we first of all find

$$\frac{\partial L}{\partial t}=0 \quad \text{and} \quad \frac{\partial L}{\partial \dot{t}}=\left(1-\frac{2M}{r}\right)\dot{t} \quad \Rightarrow \quad \frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)\dot{t}\right]=0$$

Taking *a=2* we then get

$$\frac{\partial L}{\partial \theta}=-r^2\sin\theta\cos\theta\ \dot{\phi}^2 \quad \text{and} \quad \frac{\partial L}{\partial \dot{\theta}}=-r^2\dot{\theta}$$

$$\Rightarrow \quad \frac{d}{d\tau}\left(-r^2\dot{\theta}\right)+r^2\sin\theta\cos\theta\ \dot{\phi}^2=0$$

Finally, using *a=3* we find

$$\frac{\partial L}{\partial \phi}=0 \quad \text{and} \quad \frac{\partial L}{\partial \dot{\phi}}=-r^2\sin^2\theta\ \dot{\phi} \quad \Rightarrow \quad \frac{d}{d\tau}\left(-r^2\sin^2\theta\ \dot{\phi}\right)=0$$

These are 3 of the 4 equations that we need...

To determine the final equation we can either set *a=1* or simply use the fact that we must have *2L=1* for a timelike geodesic. This leads to;

$$\left(1-\frac{2M}{r}\right)\dot{t}^2-\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2-r^2\left(\dot{\theta}^2+\sin^2\theta\ \dot{\phi}^2\right)=1$$

Equatorial orbits

We want to understand what these equations are telling us.

Begin by considering a geodesic in the equatorial plane, θ=π/2.

From the 2nd of our equations we then see that

$$\frac{d}{d\tau}\left(-r^2\dot{\theta}\right)+r^2\sin\theta\cos\theta\;\dot{\phi}^2=0\quad\Rightarrow\quad -r^2\ddot{\theta}-2r\dot{r}\dot{\theta}=0$$

Noting that one can always rotate the axes in such a way that, at *t=0* one has $\dot{\theta}=0$, we realise that we can show that <u>all</u> derivatives vanish.

In other words, these geodesics will <u>remain</u> in the equatorial plane.

Now consider the 3rd equation;

$$\frac{d}{d\tau}\left(r^2\dot{\phi}\right)=0\quad\Rightarrow\quad r^2\dot{\phi}=J=\text{constant}$$

This equation represents the <u>conservation of angular momentum</u> *J*.

The 1st equation leads to another constant of motion;

$$\frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)\dot{t}\right]=0 \quad \Rightarrow \quad \left(1-\frac{2M}{r}\right)\dot{t}=E=\text{constant}$$

The <u>energy</u>, E, of the orbit is conserved.

Using these definitions in the radial (4th) equation we get

$$\left(1-\frac{2M}{r}\right)^{-1}E^2-\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2-\frac{J^2}{r^2}=1$$

or

$$\dot{r}^2=E^2-\left(1-\frac{2M}{r}\right)\left(1+\frac{J^2}{r^2}\right)$$

We have arrived at a "final" equation for equatorial (timelike) geodesics;

$$\dot{r}^2 + \left(1 - \frac{2M}{r}\right)\left(1 + \frac{J^2}{r^2}\right) = E^2$$

What does this equation tell us?

Taking a derivative with respect to τ we get

$$\ddot{r} + \frac{M}{r^2} - \frac{J^2}{r^3} + \frac{3MJ^2}{r^4} = 0$$

This shows that, for circular orbits with $r=R=$constant, we will have

$$R = \frac{J^2}{2M}\left[1 \pm \left(1 - \frac{12M^2}{J^2}\right)^{1/2}\right]$$

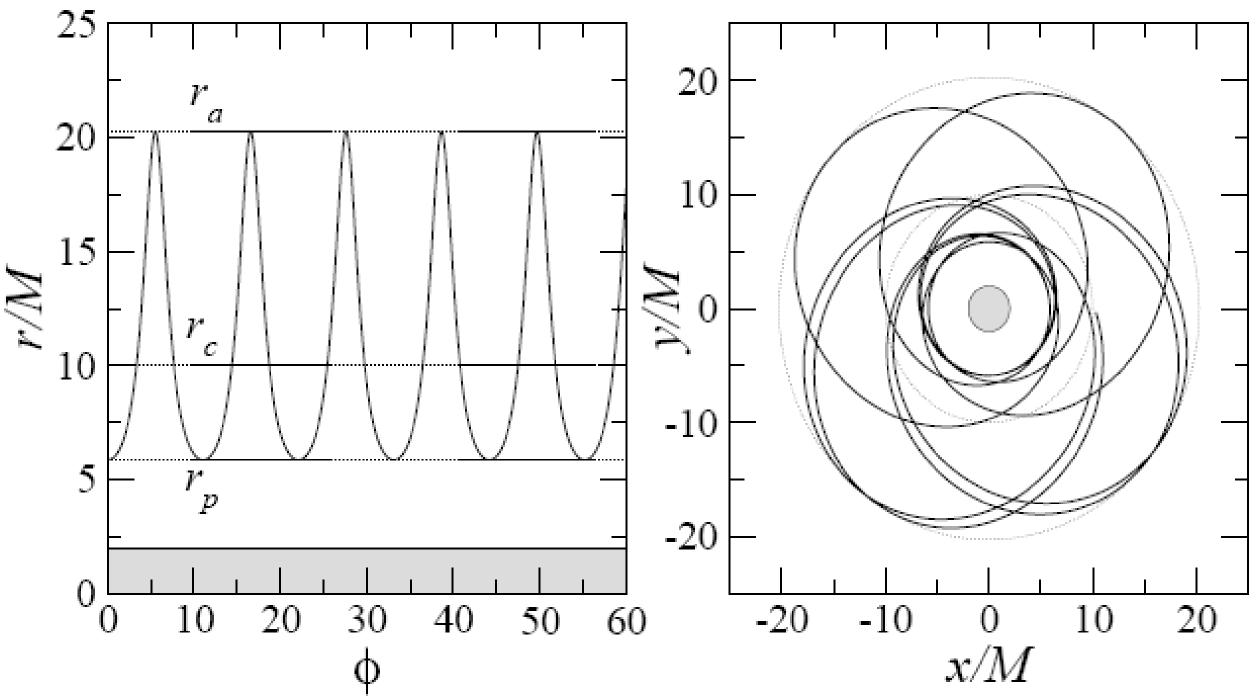
We learn that, for each value of $J^2 \geq 12M^2$ there is a pair of circular orbits. However, a detailed analysis shows that only the outer one is stable.

The innermost stable circular orbit corresponds to $R=6M$.

Non-circular orbits can be quite complicated.

Bound orbits have two “turning points”, in the solar system known as perihelion and aphelion.

The perihelion “advances” with each completed orbit.



Two-body problem

Let us consider the equatorial orbits from a different point of view.

We had the two equations;

$$r^2\dot{\phi} = J$$

and

$$\dot{r}^2 + \left(1 - \frac{2M}{r}\right)\left(1 + \frac{J^2}{r^2}\right) = E^2$$

Let us re-write the latter in terms of a new variable $u=1/r$ where $u=u(\phi)$.

To do this, we need

$$\dot{r} = \frac{dr}{du}\frac{du}{d\tau} = -\frac{1}{u^2}\frac{du}{d\tau} = -r^2\frac{du}{d\phi}\frac{d\phi}{d\tau} = -J\frac{du}{d\phi}$$

and we arrive at

$$J^2\left(\frac{du}{d\phi}\right)^2 + \left(1 - 2Mu\right)\left(1 + J^2u^2\right) = E^2$$

This equation can be written

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{E^2 - 1}{J^2} + \frac{2M}{J^2}u + 2Mu^3$$

Taking a derivative of this we arrive at

$$\underbrace{\frac{d^2u}{d\phi^2} + u = \frac{M}{J^2}}_{\text{Binet's equation}} + 3Mu^2$$

This equation is very similar to the one that describes the two-body problem in Newtonian gravity (Binet's equation). In addition, we have a term representing the general relativistic correction.

In analysing this problem, it is helpful to introduce a dimensionless parameter $\varepsilon = 3M^2/J^2$ such that

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{J^2} + \varepsilon\left(\frac{J^2u^2}{M}\right)$$

In order to solve

$$\frac{d^2u}{d\phi^2} + u = \frac{M}{J^2} + \varepsilon\left(\frac{J^2u^2}{M}\right)$$

we note that $J^2 \geq 12M^2 \;\rightarrow\; \varepsilon \leq 1/4$

Hence, it makes sense to use a perturbative method. Take

$$u = u_0 + \varepsilon u_1 + O\left(\varepsilon^2\right)$$

to get

$$\underbrace{\left(\frac{d^2u_0}{d\phi^2} + u_0 - \frac{M}{J^2}\right)}_{=0} + \varepsilon\underbrace{\left(\frac{d^2u_1}{d\phi^2} + u_1 - \frac{J^2u_0^2}{M}\right)}_{=0} + O\left(\varepsilon^2\right) = 0$$

The Newtonian problem

$$\frac{d^2u_0}{d\phi^2} + u_0 = \frac{M}{J^2}$$

has solution

$$u_0 = \frac{M}{J^2}\left[1 + e\cos\left(\phi - \phi_0\right)\right]$$

Or, if we orient the axes so that $\phi_0 = \phi(t=0) = 0$;

$$u_0 = \frac{M}{J^2}\left(1 + e\cos\phi\right)$$

The orbit is an ellipse.

To solve for the relativistic corrections, we need

$$\frac{J^2 u_0^2}{M} = \frac{M}{J^2}\left(1+e\cos\phi\right)^2 = \frac{M}{J^2}\left(1+\tfrac{1}{2}e^2\right) + \frac{2Me}{J^2}\underbrace{\cos\phi}_{\text{hom. sol!}} + \frac{Me^2}{2J^2}\cos 2\phi$$

The standard method leads to the solution to the order ε problem;

$$u_1 = \frac{M}{J^2}\left(1+\tfrac{1}{2}e^2\right) + \frac{Me}{J^2}\underbrace{\phi\sin\phi}_{\text{secular!}} - \frac{Me^2}{6J^2}\cos 2\phi$$

The magnitude of one of these terms increases in time. It will soon grow to be the most important. Hence, we can approximate the total solution as

$$u \approx u_0 + \varepsilon\frac{Me}{J^2}\phi\sin\phi \approx \frac{M}{J^2}\left\{1+e\cos\left[\phi\left(1-\varepsilon\right)\right]\right\} + O\left(\varepsilon^2\right)$$

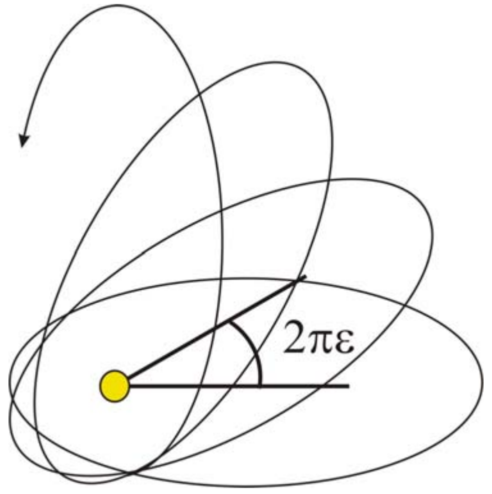
This shows that the period is not 2π, but

$$\frac{2\pi}{1-\varepsilon} \approx 2\pi\left(1+\varepsilon\right)$$

Perihelion advance of Mercury

The effect that we have determined means that the elliptical orbits do not quite close.

With each revolution, the perihelion advances by $2\pi\varepsilon$.



The effect is larger for bodies in closer orbits. In the solar system, the largest predicted effect is for Mercury.

Einstein worked out that Mercury's perihelion should advance by 43 arcseconds per century.

This agrees very well with observations, and is one of the classic tests of Einstein's theory.

Note: The presence of other bodies (like Jupiter) leads to a much larger effect on Mercury's orbit (5601 arcseconds per century).