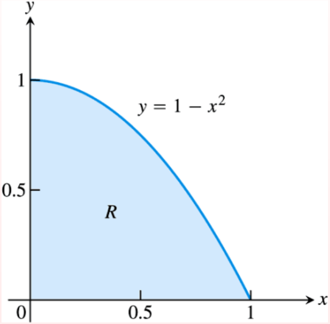
MTH4100 Calculus I

Bill Jackson
School of Mathematical Sciences QMUL

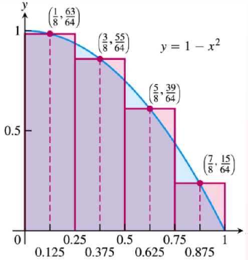
Week 9, Semester 1, 2012

Example:



How can we compute the area of the shaded region R?

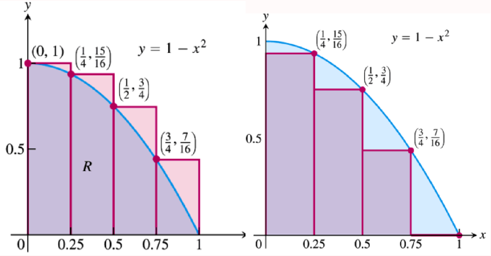
Approximation algorithm



- Subdivide the interval $[0, 1]$ into n subintervals of equal width $\Delta x = \frac{1}{n}$.
- Choose a point c_k in the k'th subinterval. For example we could use:
  1. *midpoint rule:* Choose c_k in the *middle* of the k'th subinterval.
  2. *max rule:* Choose c_k such that $f(c_k)$is *maximum*.
  3. *min rule:* choose c_k such that $f(c_k)$ is *minimum*.

Approximation algorithm - continued

- Construct n rectangles with base Δx and height c_k.



- Approximate the area by calculating the sum of the areas of these rectangles $f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_n)\Delta x$.

Approximation algorithm - continued

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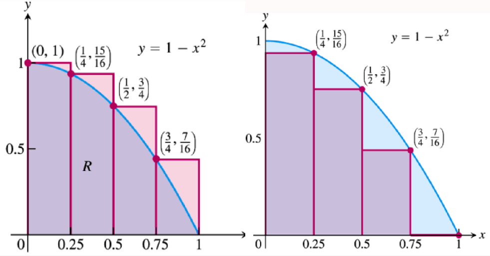

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Note that the area of R will lie between the *upper sum* i.e the sum we obtain using the max rule to choose the points c_k and the *lower sum* i.e the sum we obtain using the min rule to choose the points c_k. So we can estimate how close our approximation is to the correct area by calculating the difference between these two sums.

Approximation algorithm - continued

We can improve the accuracy of our approximation by choosing shorter subintervals i.e. larger values for n.

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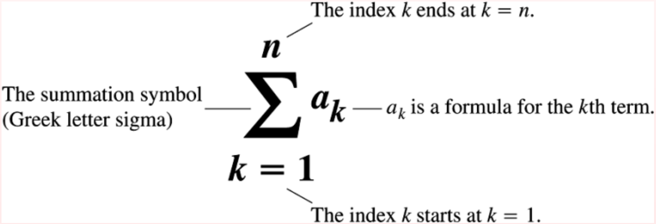
If the approximations converge to the same limit as $n \to \infty$, *no matter how we choose the points* c_k, then this limit will be the area of R.

Finite sums

To handle sums with many terms, we need a more concise notation. Let

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

where



Example: $\sum_{k=1}^{3} (-1)^k k$

Algebra Rules for Finite Sums

1. *Sum Rule*: $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. *Difference Rule*: $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
3. *Constant Multiple Rule*: $\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$ (Any number c)
4. *Constant Value Rule*: $\sum_{k=1}^{n} c = n \cdot c$ (c is any constant value.)

Example:

$$\sum_{k=1}^{n}(5k - k^3) = 5\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^3$$

by rules 1 and 2.

Some common sums

Theorem

- *Sum of first n natural numbers:*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

- *Sum of first n squares:*

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

- *Sum of first n cubes:*

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Example continued

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- Subdivide the interval into n subintervals of width $\Delta x = \frac{1}{n}$:
  $\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{3}{n}\right], \ldots, \left[\frac{n-1}{n}, \frac{n}{n}\right].$

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- Use the min rule to choose the points c_k: this gives $c_k = \frac{k}{n}$, $k \in \mathbb{N}$ is the rightmost point in the k'th subinterval.

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- Use the min rule to choose the points c_k: this gives $c_k = \frac{k}{n}$, $k \in \mathbb{N}$ is the rightmost point in the k'th subinterval.
- The lower sum is
  $f\left(\frac{1}{n}\right)\frac{1}{n} + f\left(\frac{2}{n}\right)\frac{1}{n} + \ldots + f\left(\frac{n}{n}\right)\frac{1}{n} = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$.
  Hence the area of R is at least $\frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$.

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- As $n \to \infty$, both sums converge to $\frac{2}{3}$. Therefore, the area of R is $\frac{2}{3}$.

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- As $n \to \infty$, both sums converge to $\frac{2}{3}$. Therefore, the area of R is $\frac{2}{3}$.

Note that any other choice of c_k would give the same limit (since its sum must lie between the upper and lower sums).

Lecture continued on whiteboard - see online lecture notes.