MTH4100 Calculus I

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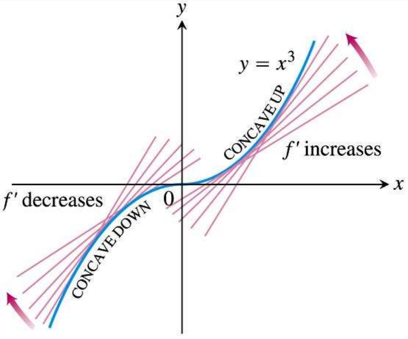
DEFINITION **Concave Up, Concave Down**

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I

(b) **concave down** on an open interval I if f' is decreasing on I.



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Example: $y = x^{1/3}$.

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Example $y = x^4$.

If f is a function, c is a critical point of f and f is twice differentiable at c then we can use the second derivative $f''(c)$ to test whether $f(c)$ is a local extremum of f:

Second derivative test for local extrema

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Theorem

Suppose f is a function, $f'(c) = 0$, and f'' is continuous on some open interval which contains c.

1. *If $f''(c) < 0$ then f has a local maximum at c.*
2. *If $f''(c) > 0$ then f has a local minimum at c.*
3. *If $f''(c) = 0$ then the test fails, f can have either a local maximum, a local minimum, or a point of inflection at c.*

Step 1 Identify the natural domain of f and find any symmetries the graph may have.

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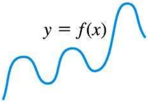
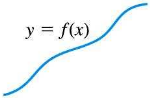
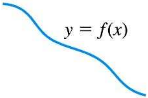
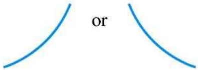
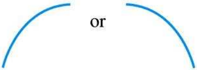
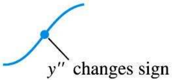
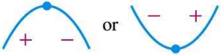
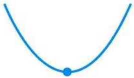
Example Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

Summary: Learning about functions from derivatives

 Differentiable $\Rightarrow$ smooth, connected; graph may rise and fall	 $y' > 0 \Rightarrow$ rises from left to right; may be wavy	 $y' < 0 \Rightarrow$ falls from left to right; may be wavy
 $y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall	 $y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall	 Inflection point
 y' changes sign $\Rightarrow$ graph has local maximum or local minimum	 $y' = 0$ and $y'' < 0$ at a point; graph has local maximum	 $y' = 0$ and $y'' > 0$ at a point; graph has local minimum