MTH4100 Calculus I

Lecture notes for Week 9

Thomas' Calculus, Sections 4.4, 7.5, 4.7, and 5.1 to 5.3

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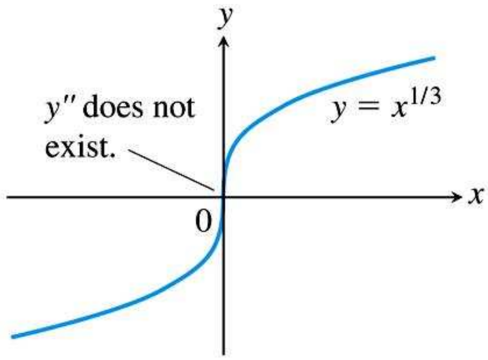
Points of inflection

We have seen that the graph of $y = x^3$ changes concavity at the point $(0, 0)$. Such a point is covered by the following definition.

> **DEFINITION Point of Inflection**
> A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

The condition that the graph of the function has a tangent line at a point is more general than saying that the function is differentiable at the point since it allows the tangent line to be vertical (and hence the derivative to be 'infinite').

Example Consider $y = x^{1/3}$. We have $y' = \frac{1}{3}x^{-\frac{2}{3}}$ and $y'' = -\frac{2}{9}x^{-\frac{5}{3}}$. Hence y'' does not exist at $x = 0$. On the other hand $\lim_{x\to 0^-} y''(x) = \infty$ and $\lim_{x\to 0^+} y''(x) = -\infty$. Thus y'' changes sign as we pass through $x = 0$ and we do have a point of inflection at $x = 0$ (even though $y''(0)$ does not exist).

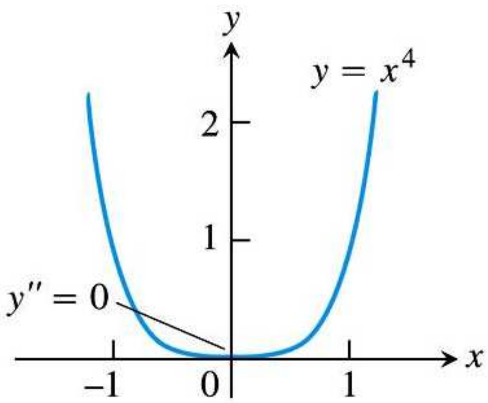


Suppose f is a function. At a point of inflection $(c, f(c))$ of f we have $f''(x) > 0$ on one side of c, $f''(x) < 0$ on the other side of c, and either $f''(c) = 0$ or f'' is undefined at c itself. Thus, if $f''(c)$ exists, then $(c, f(c))$ is a point of inflection if and only if $f''(c) = 0$ AND f' has a local maximum or minimum at $x = c$.

Example: Consider $f(x) = x^3 - 3x$. We have $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. Since $f''(0) = 0$ and $f'(0) = -3$ is a local minimum of f', the graph of f has a point of inflection at $x = 0$.

Note, however, that we can have $f''(c) = 0$ without $(c, f(c))$ being a point of inflection (when f' does not change sign at $x = c$).

Example Consider $y = x^4$. We have $y' = 4x^3$ and $y'' = 12x^2$. Thus $y''(0) = 0$. BUT y'' does not change sign at $x = 0$. Hence there is no inflection point at $x = 0$.

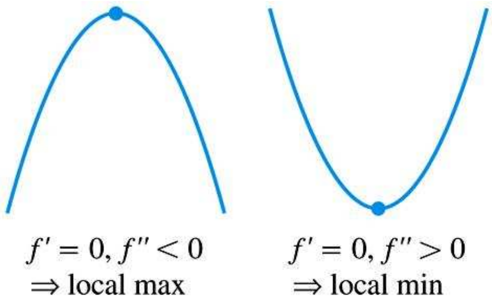


If f is a function, c is a critical point of f and f is twice differentiable at c then we can use the second derivative $f''(c)$ to test whether $f(c)$ is a local extremum of f:

Theorem 1 (Second derivative test for local extrema) *Suppose f is a function, $f'(c) = 0$, and f'' is continuous on some open interval which contains c.*

1. *If $f''(c) < 0$ then f has a local maximum at c.*
2. *If $f''(c) > 0$ then f has a local minimum at c.*
3. *If $f''(c) = 0$ then the test fails, f can have either a local maximum, a local minimum, or a point of inflection at c.*

Proof Suppose $f''(c) < 0$. Then f' is decreasing at c. Since $f'(c) = 0$, f' must change sign from $+$ to $-$ as we pass through c. Hence f has a local maximum at c. A similar proof holds if $f''(c) > 0$.



If $f''(c) = 0$ then either the sign of f' changes as we pass through c and f has a local extramum at c (e.g. $f(x) = x^4$), or the sign of f' does not change as we pass through c and f has a point of inflection at c (e.g. $f(x) = x^3$). •

Note. In case (3) we can use the first derivative test to determine if f has a local extremum or a point of inflection at c.

Graph Drawing: Strategy for Graphing $y = f(x)$

Step 1 Identify the natural domain of f and find any symmetries the graph may have.

Step 2 Determine f' and f''.

Step 3 Find the critical points of f and determine the functions behavior at each one.

Step 4 Determine where f is increasing or decreasing.

Step 5 Find the points of inflection, if any occur, and determine where the graph is concave up or concave down.

Step 6 Investigate the behavior of $f(x)$ as $x \to \pm\infty$ and identify any asymptotes.

Step 7 Plot key points such as the intercepts of the graph on the axes and the points found in Steps 3-5, then sketch the graph.

Example: Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

Step 1 The natural domain of f is $(-\infty, \infty)$. There are no symmetries about any axis.

Step 2

$$f'(x) = \frac{2(1-x^2)}{(1+x^2)^2}$$

$$f''(x) == \frac{4x(x^2-3)}{(1+x^2)^3}$$

Step 3 $f'(x)$ exists for all $x \in (-\infty, \infty)$ and $f(x) = 0$ when $x = \pm 1$, so $x = -1$ and $x = 1$ are the only critical points of f. We have $f''(-1) = 1 > 0$ and $f''(1) = -1 < 0$ so $(-1, 0)$ is a local minimum and $(1, 2)$ a local maximum.

Step 4 We have: $f'(x) < 0$ for $x \in (-\infty, -1)$ so the curve is decreasing on $(-\infty, -1)$; $f'(x) > 0$ for $x \in (-1, 1)$ so the curve is increasing on $(-1, 1)$; $f'(x) < 0$ for $x \in (1, \infty)$ so the curve is decreasing on $(1, \infty)$;

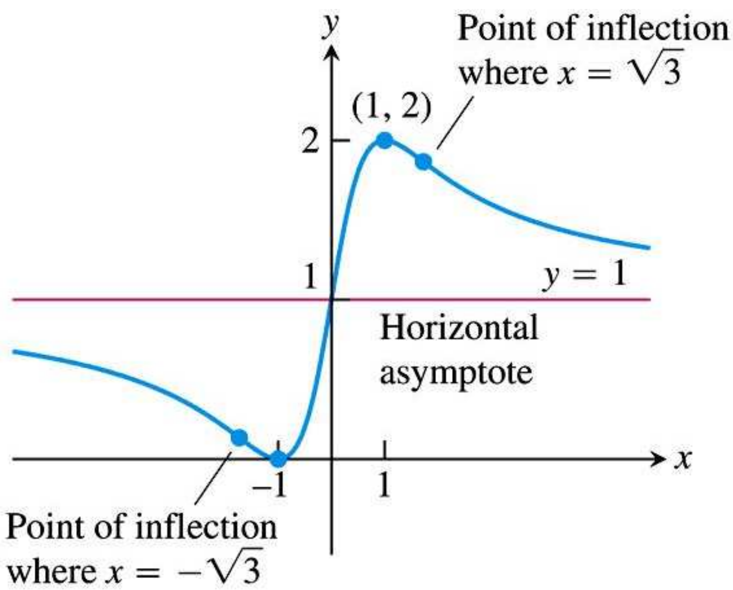
Step 5 $f''(x) = 0$ when $x = \pm\sqrt{3}$ or 0; $f'' < 0$ on $(-\infty, -\sqrt{3})$ so graph is concave down; $f'' > 0$ on $(-\sqrt{3}, 0)$ so graph is concave up; $f'' < 0$ on $(0, \sqrt{3})$ so graph is concave down; $f'' > 0$ on $(\sqrt{3}, \infty)$ so graph is concave up. Each point is a point of inflection.

Step 6

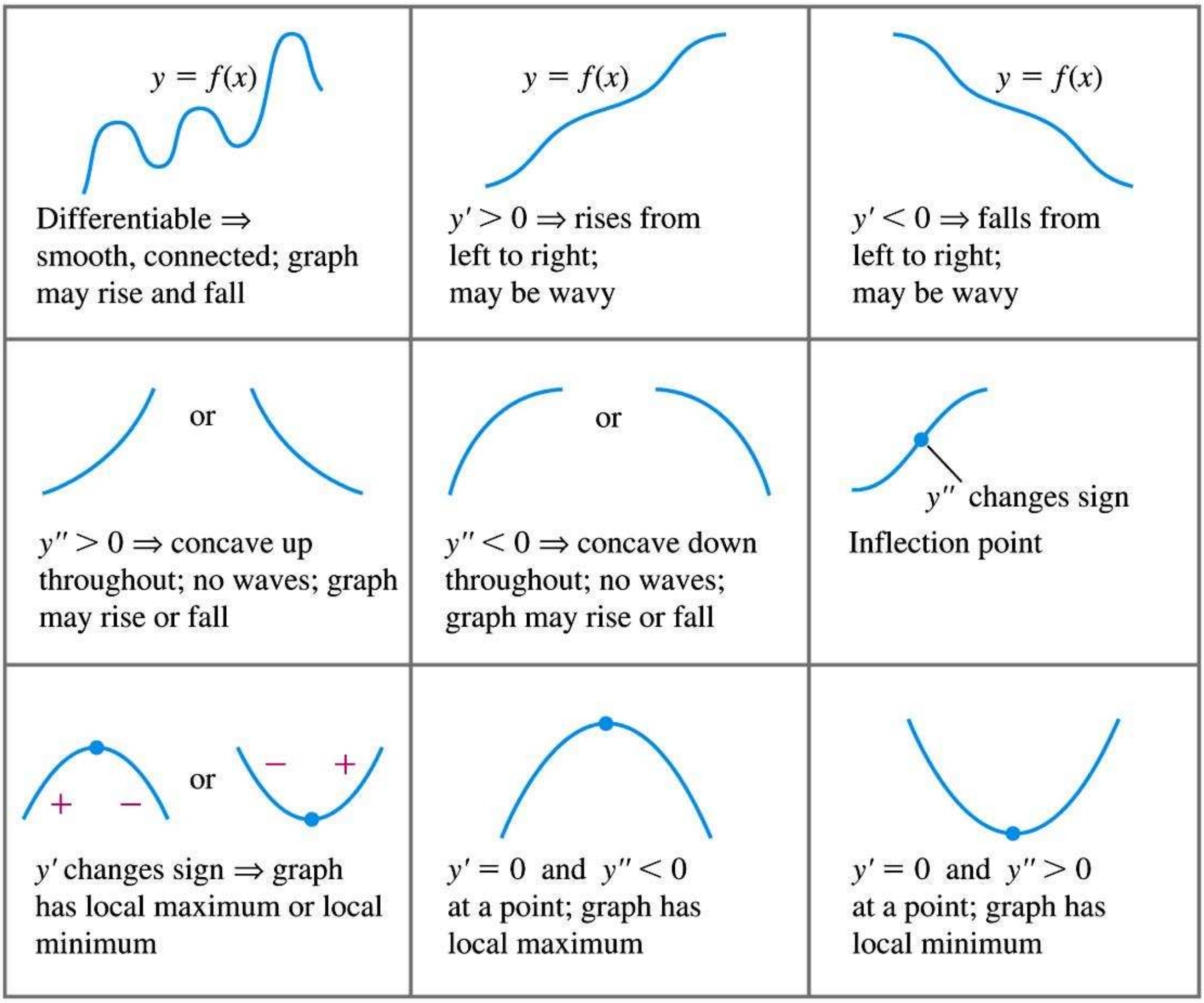
$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} = \frac{1+2/x+1/x^2}{1/x^2+1}$$

so $\lim_{x\to-\infty} f(x) = 1 = \lim_{x\to\infty} f(x)$ so $y = 1$ is a horizontal asymptote. There are no vertical asymptotes.

Step 3 $f(0) = 1$ and $f(x) = 0$ when $x = -1$ so the graph meets the y-axis at $y = 1$ and the x-axis at $x = -1$. We can now sketch the curve:



Summary: Learning about functions from derivatives



L'Hôpital's Rule and Indeterminate Forms

If $f(a) = g(a) = 0$, $f(a)/g(a) = 0/0$ is a meaningless expression, called an *indeterminate form*. In this case $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ cannot be found by simply substituting $x = a$. L'Hôpital's Rule gives us a method to calculate this limit if f and g are both differentiable at $x = a$.

Theorem 2 (L'Hôpital's Rule - Weak Form)) *Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ both exist, and that $g'(a) \neq 0$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} .$$

Proof We have

$$\begin{aligned}
\frac{f'(a)}{g'(a)} &= \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} && \text{(definition of } f',g')\\
&= \lim_{x\to a}\left(\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}\right) && \text{(limit laws)}\\
&= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}\\
&= \lim_{x\to a}\frac{f(x)}{g(x)} && \text{(since } f(a)=0=g(a))
\end{aligned}$$

•

Example: $\lim\limits_{x\to 0}\dfrac{5x-\sin x}{x} = \dfrac{5-\cos 0}{1} = 4.$

WARNING: Always check for "0/0", i.e., $f(a)=g(a)=0$, before using l'Hôpital. Otherwise you may get a wrong answer.

Example: We have $\lim\limits_{x\to 0}\dfrac{1+\sin x}{1-x} = \dfrac{1}{1} = 1$ by direct substitution. BUT if we tried to use l'Hôpital we would get $\lim\limits_{x\to 0}\dfrac{1+\sin x}{1-x} = \dfrac{\cos 0}{-1} = -1.$

Sometimes we have to use l'Hôpital's rule more than once. For example if we try to calculate $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}$ by differentiating the numerator and denominator and then substituting $x=0$, we still obtain $0/0$. To apply l'Hôpital's rule again we need a stronger version of the rule:

Theorem 3 (L'Hôpital's Rule - Strong Form)) *Suppose that $f(a)=g(a)=0$, that f and g are differentiable on an open interval I containing a, and that $g'(x)\neq 0$ on I* **if** *$x\neq a$. Then*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}\ ,$$

assuming that the limit on the right side exists.

See textbook Section 7.5 for a proof.

Example: We have

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{1-\cos x}{3x^2} = \lim_{x\to 0}\frac{\sin x}{6x} = \lim_{x\to 0}\frac{\cos x}{6} = \frac{1}{6}\ .$$

Summary:

> **Using L'Hôpital's Rule**
> To find
> $$\lim_{x \to a} \frac{f(x)}{g(x)}$$
> by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Remark: L'Hôpital also applies to one-sided limits.

Example:

$$\lim_{x \to 0^+} \frac{\sin x}{x^2} = \lim_{x \to 0^+} \frac{\cos x}{2x} = \infty$$

and

$$\lim_{x \to 0^-} \frac{\sin x}{x^2} = \lim_{x \to 0^-} \frac{\cos x}{2x} = -\infty$$

What about limits involving other indeterminate forms such as ∞/∞, $\infty \cdot 0$ or $\infty - \infty$?

$\boldsymbol{\infty/\infty}$: It can be shown that if $\lim_{x \to a} f(x) = \infty = \lim_{x \to a} g(x)$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

So we can use L'Hôpital rule in the same way as for "0/0".
Example:

$$\lim_{x \to \infty} \frac{x - x^2}{x^2 + 7x} = \lim_{x \to \infty} \frac{1 - 2x}{2x + 7} = \lim_{x \to \infty} \frac{-2}{2} = -1$$

$\boldsymbol{\infty \cdot 0}$: Use

$$\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} \frac{g(x)}{1/f(x)}$$

Example:

$$\lim_{x \to \infty} x \sin(1/x) = \lim_{x \to \infty} \frac{\sin(1/x)}{1/x} = \lim_{x \to \infty} \frac{(-1/x^2)\cos(1/x)}{-1/x^2} = \lim_{x \to \infty} \cos(1/x) = 1.$$

$\boldsymbol{\infty - \infty}$: Try to gather terms so we can use the standard form of L'Hôpital rule:
Example:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} = \lim_{x \to 0} \frac{\sin x}{2\cos x - x \sin x} = 0$$

Antiderivatives

Idea: Given a function f, find a function F such that $F' = f$.

DEFINITION Antiderivative
A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Examples: (1) If $f(x) = 2x$ then we can take $F(x) = x^2$.
(2) If $h(x) = \sin x$ then we can take $H(x) = -\cos x$.

It is easy to see that if $F(x)$ is an antiderivative of $f(x)$ then $F(x)+C$ will be an antiderivative of $f(x)$ for any constant $C \in \mathbb{R}$. Furthermore, if $G(x)$ is any other antiderivative of $f(x)$ then we have $F'(x) = f(x) = G'(x)$ and the second corollary to the Mean Value Theorem tells us that $G(x) = F(x) + C$ for any constant $C \in \mathbb{R}$. This gives:

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Some antiderivative formulas are shown in the following table:

TABLE 4.2 Antiderivative formulas

	Function	General antiderivative
1.	x^n	$\dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1$, n rational
2.	$\sin kx$	$-\dfrac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\dfrac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

Examples: (1) $f(x) = x^4 \Rightarrow F(x) = \frac{x^5}{5} + C$

(2) $h(x) = \cos 5x \Rightarrow H(x) = \frac{\sin 5x}{5} + C$

The following result can easily be verified by differentiating each of the antiderivatives:

Lemma 1 (Antiderivative linearity rules) *Suppose $f(x), g(x)$ are functions with antiderivatives $F(x)$ and $G(x)$, and $k \in \mathbb{R}$. Then:*

- *$kf(x)$ has general antiderivative $kF(x) + C$;*
- *$f(x) + g(x)$ has general antiderivative $F(x) + G(x) + C$;*

for any constant $C \in \mathbb{R}$.

Example: Find the general antiderivative of $h(x) = \dfrac{5}{\sqrt{x}} + \sin 3x$.

- We have $h(x) = 5f(x) + g(x)$ with $f(x) = x^{-1/2}$ and $g(x) = \sin 3x$.
- $F(x) = 2\sqrt{x} + C_1$, which satisfies $F'(x) = f(x)$.
- $G(x) = -\frac{1}{3}\cos 3x + C_2$, which satisfies $G'(x) = g(x)$.
- Therefore
$$H(x) = 10\sqrt{x} - \frac{1}{3}\cos 3x + C \ , \ \text{where } C = C_1 + C_2 \ .$$

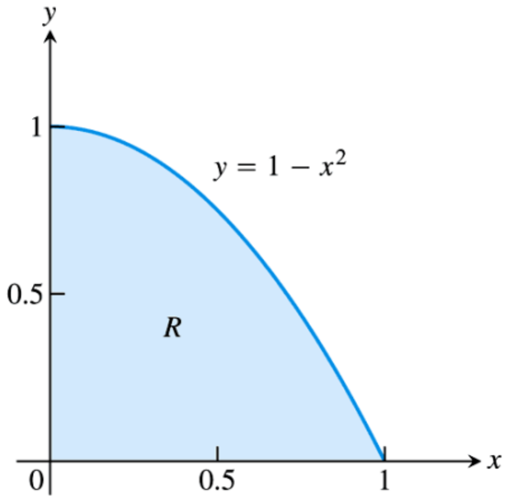
Definition We refer to the general antiderivative, $F(x)+C$, of $f(x)$ as the *indefinite integral* of $f(x)$ and denote it by
$$\int f(x)dx \, .$$
The symbol $\int$ is called an *integral sign*, the function f is the *integrand* and the variable x is the *variable of integration.*
Examples:

1. $\int 4x\,dx = 2x^2 + C$
2. $\int \cos x\,dx = \sin x + C$

Integration

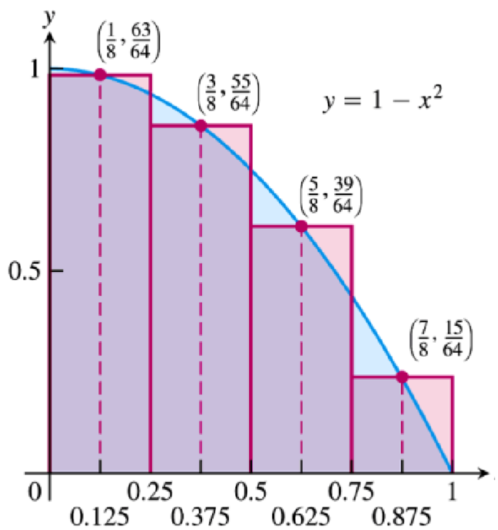
Estimating areas with finite sums

Example:



How can we compute the area or the shaded region R?

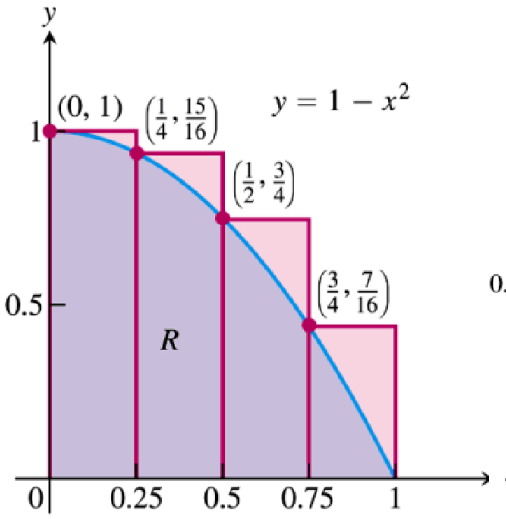
Approximation algorithm

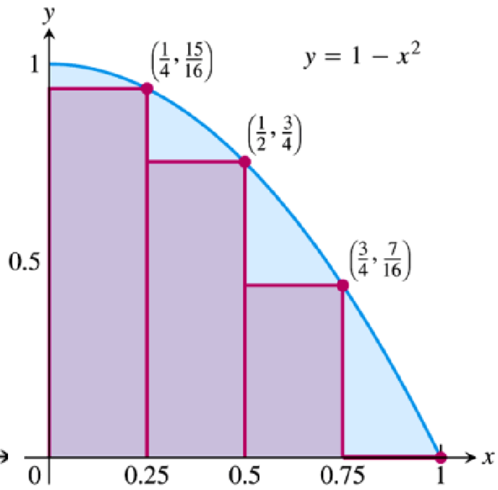


- Subdivide the interval $[0,1]$ into n subintervals of equal width $\Delta x = \frac{1}{n}$.
- Choose a point c_k in the k'th subinterval. For example we could use

  1. *midpoint rule:* Choose c_k in the *middle* of the k'th subinterval.
  2. *max rule:* Choose c_k such that $f(c_k)$is *maximum*.
  3. *min rule:* choose c_k such that $f(c_k)$ is *minimum*.

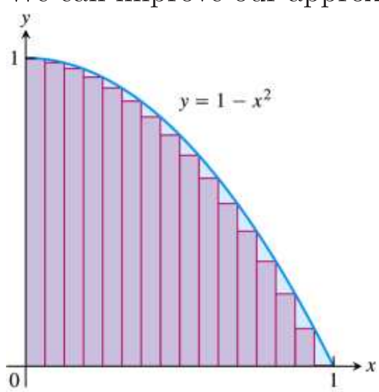
- Construct n rectangles with base Δx and height c_k.





- Approximate the area by calculating the sum $f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_n)\Delta x$.

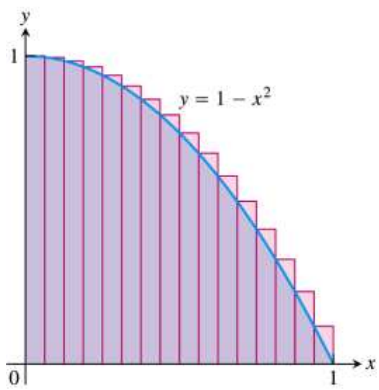
Note that the area of R will lie between the *upper sum* i.e the sum we obtain using the max rule to choose the points c_k and the *lower sum* i.e the sum we obtain using the min rule to choose the points c_k. So we can estimate how close our approximation is to the correct area by calculating the difference between these two sums.
We can improve our approximation by choosing shorter subintervals i.e. larger values for n:



(a)

TABLE 5.1 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint rule	Upper sum
2	.375	.6875	.875
4	.53125	.671875	.78125
16	.634765625	.6669921875	.697265625
50	.6566	.6667	.6766
100	.66165	.666675	.67165
1000	.6661665	.66666675	.6671665



(b)

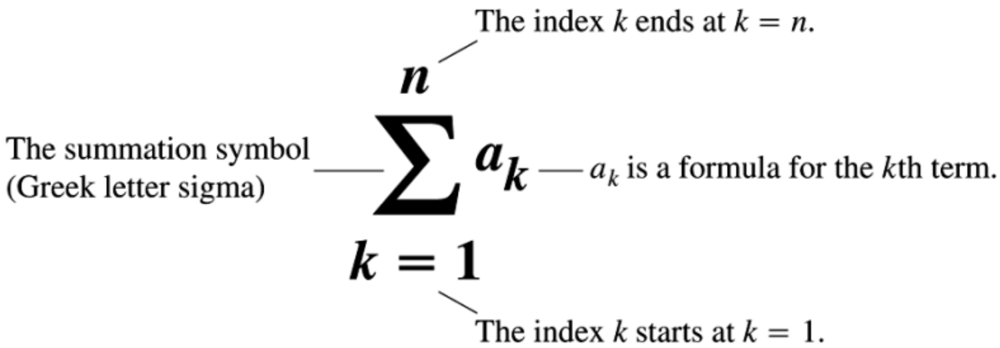
If these approximations converge to the same limit as $n \to \infty$, *no matter how we choose the points* c_k, then this limit will be the area of R.

Finite sums

To handle sums with many terms, we need a more concise notation. Let

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

where



Examples: (1) $f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_n)\Delta x = \sum_{k=1}^{n} f(c_k)\Delta x$

(2) $\sum_{k=1}^{3}(-1)^k k = (-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3 = -1 + 2 - 3 = -2$

(3) $1+3+5+7+9 = \sum_{k=1}^{5}(2k-1)$. Note however that the same sum can be expressed in many ways by changing the index of summation. We have $1+3+5+7+9 = \sum_{n=0}^{4}(2n+1)$ and $1+3+5+7+9 = \sum_{x=-3}^{1}(2x+7)$.

Algebra Rules for Finite Sums

1.	*Sum Rule*:	$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$	
2.	*Difference Rule*:	$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$	
3.	*Constant Multiple Rule*:	$\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$	(Any number c)
4.	*Constant Value Rule*:	$\sum_{k=1}^{n} c = n \cdot c$	(c is any constant value.)

Example: $\sum_{k=1}^{n}(5k - k^3) = 5\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^3$ by rules 1 and 2.

Can we calculate these sums?

Theorem 4 • *Sum of first n natural numbers:*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

- *Sum of first n squares:*
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$
- *Sum of first n cubes:*
$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

This result can be proved by mathematical *induction*, see textbook Appendix 2.

Example continued

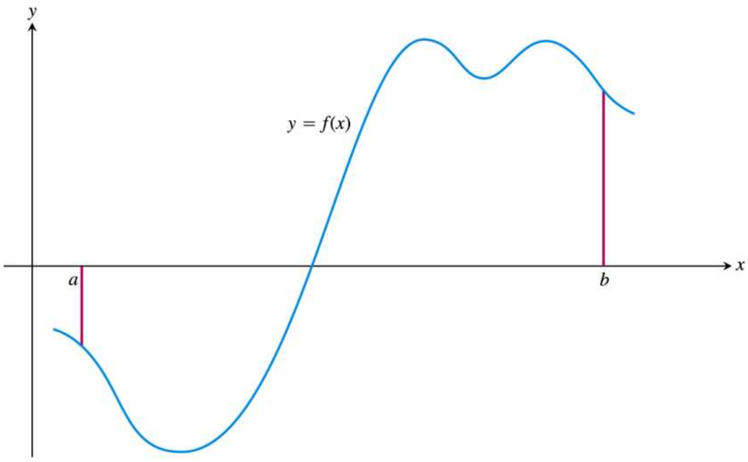
We can now compute the area of the region R below the graph of $y = 1 - x^2$ and above the interval $[0, 1]$.

- Subdivide the interval into n subintervals of width $\Delta x = \frac{1}{n}$:
$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{3}{n}\right], \ldots, \left[\frac{n-1}{n}, \frac{n}{n}\right].$$
- Use the min rule to choose the points c_k: this gives $c_k = \frac{k}{n}$, $k \in \mathbb{N}$ is the rightmost point in the k'th subinterval.
- The lower sum is
$$\begin{aligned} f\left(\frac{1}{n}\right)\frac{1}{n} + f\left(\frac{2}{n}\right)\frac{1}{n} + \ldots + f\left(\frac{n}{n}\right)\frac{1}{n} &= \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} \\ &= \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n}\right)^2\right)\frac{1}{n} \\ &= \sum_{k=1}^{n} \left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\ &= \frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\ &= \frac{1}{n}n - \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \\ &= 1 - \frac{2n^2+3n+1}{6n^2} \\ &= \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}. \end{aligned}$$
Hence the area of R is at least $\frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$.
- A similar calculation shows that the upper sum is $\frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}$ and hence the area of R is at most $\frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$.
- As $n \to \infty$, both sums converge to $\frac{2}{3}$. Therefore, the area of R is $\frac{2}{3}$.

Note that any other choice of c_k would give the same limit (since the corresponding sum must lie between the upper and lower sums).

Riemann Sums and the Definite Integral

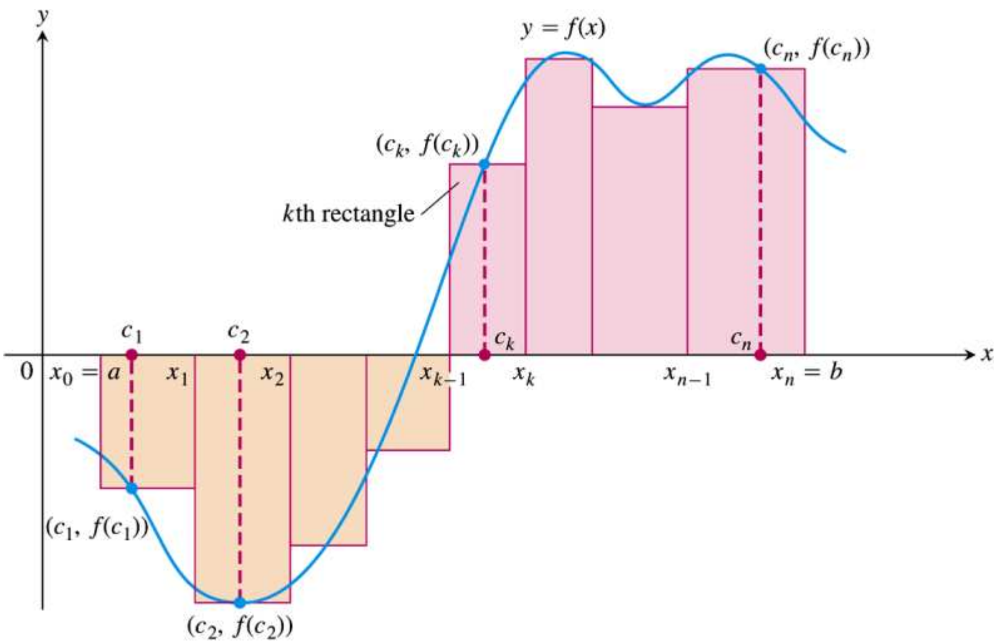
Consider a typical continuous function f over an interval $[a, b]$:



We want to calculate the area above the interval $[a, b]$ and below the curve $y = f(x)$.
We first construct a *partition* P of the interval $[a, b]$ into n subintervals by choosing $n + 1$ points $x_0, x_1, \ldots, x_n$ between a and b where

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\ .$$

The k'th subinterval of P is $[x_{k-1}, x_k]$.
Note that the width of this subinterval, $\Delta x_k = x_k - x_{k-1}$, may vary.
Choose a point $c_k \in [x_{k-1}, x_k]$ and construct the rectangles with base $[x_{k-1}, x_k]$ and height c_k:



The resulting sum

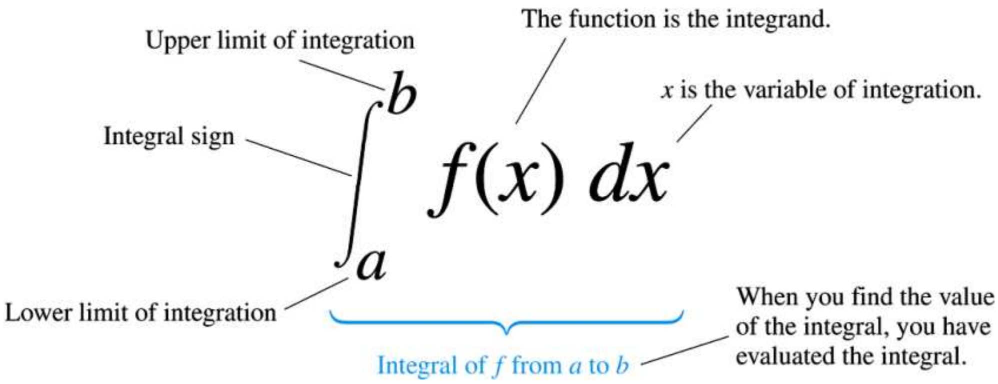
$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

is called the *Riemann sum for f on $[a,b]$ with respect to the partition P and the choice of the points c_k.*
We then choose finer and finer partitions P and consider the limits of the Riemann sums as the width of the *largest subinterval* of P goes to zero, *for all possible choices of the points c_k.* These ideas lead us to the following definition.

Definition Let f be a function defined on a closed interval $[a,b]$. We say that a real number J is the *definite integral of f over $[a,b]$* if J is the limit of all possible Riemann sums for f on $[a,b]$ as the width of the *largest subinterval* in the partition goes to zero. If such a number J exists we write

$$J = \int_a^b f(x)dx\,.$$

We have



Note that it does not matter which letter we use for the variable of integration:

$$\int_a^b f(t)dt = \int_a^b f(x)dx\,.$$