MTH4100 Calculus I

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DEFINITIONS **Absolute Maximum, Absolute Minimum**

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

These values are also called *absolute extrema*, or *global extrema*.

Extreme Value Theorem

When f is continuous and its domain is a closed interval, the existence of a global maximum and minimum is ensured by:

Theorem (Extreme Value Theorem)

If f is a continous function on a closed interval $[a, b]$, then f has both an absolute maximum value M and an absolute minimum value m. That is, there exists $x_1, x_2 \in [a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for all $x \in [a, b]$.

Local Maxima and Minima

DEFINITIONS Local Maximum, Local Minimum

A function f has a **local maximum** value at an interior point c of its domain if

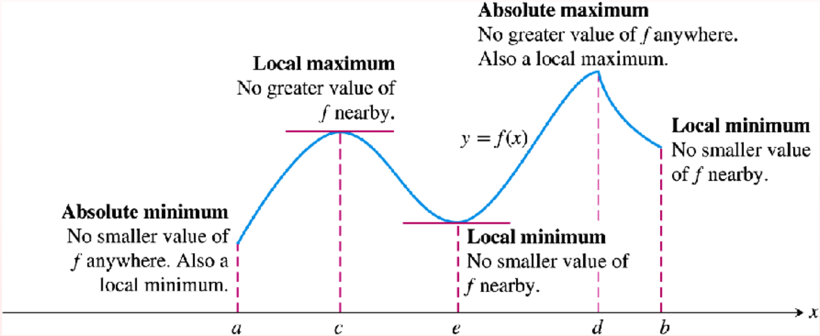
$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

These values are also called *local extrema*.

Local Maxima and Minima - Example



Note: Absolute extrema are automatically local extrema, but the converse need not be true.

First Derivative Theorem for Local Extrema

Theorem

Suppose that f has a local maximum or minimum value at an interior point c of its domain, and that f is differentiable at c. Then $f'(c) = 0$.

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This theorem tells us that the extreme values of a function f can only occur at the following kinds of points:

- interior points of the domain where $f' = 0$;
- interior points of the domain where f' does not exist;
- endpoints of the domain.

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- interior points of the domain where $f' = 0$;
- interior points of the domain where f' does not exist;
- endpoints of the domain.

Interior points of the domain of f where either $f' = 0$ or f' does not exist are called *critical points* of f.

Finding local extrema

The Extreme Value Theorem tells us that a continuous function f on a bounded closed interval has absolute maximum and minimum values. The First Derivative Theorem for Local Extrema gives us a method to determine these values:

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Examples Find the absolute extrema of:

(a) $f(x) = x^2$ on $[-1, 1]$

(b) $f(x) = x^{2/3}$ on $[-2, 3]$.

Rolle's theorem

This result tells us that a function which is continuous on a bounded closed interval and takes the same value at both endpoints of the interval must have at least one critical point in the interval.

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Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$ then there exists a $c \in (a, b)$ with $f'(c) = 0$.

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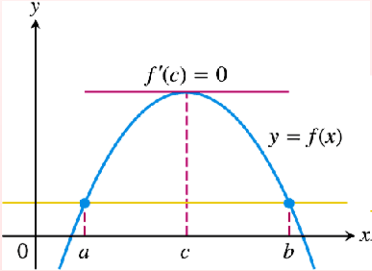

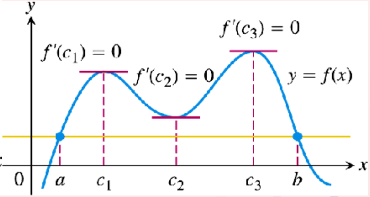

Rolle's theorem - Note

It is essential that both the hypotheses in Rolle's theorem are fulfilled i.e. f is continuous on $[a, b]$ and differentiable on (a, b):

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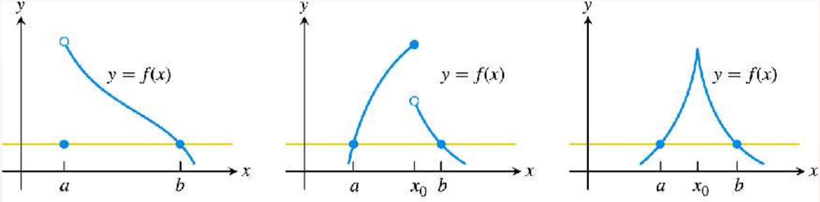

(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

In each case there is no point $c \in (a, b)$ with $f'(c) = 0$.

Rolle's theorem - Example

Apply Rolle's theorem to $f(x) = \frac{x^3}{3} - 3x$ on $[-3, 3]$.

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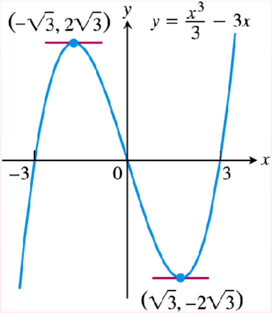

This extends Rolle's theorem to the case when $f(a) \neq f(b)$.

Theorem (Mean Value Theorem)

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}\ .$$

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Apply the Mean Value Theorem to the function $f(x) = x^2$ defined on the interval $[0, 2]$.

Corollaries to the Mean Value Theorem

Corollary (Functions with zero derivatives are constant)

If $f'(x) = 0$ *for all* $x \in (a, b)$ *then* $f(x) = C$ *for some constant* $C \in \mathbb{R}$*.*

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Corollary (Functions with the same derivative differ by a constant)

If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + C$ for some constant $C \in \mathbb{R}$.

DEFINITIONS Increasing, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

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Example: $f(x) = x^2$ *decreases* on $(-\infty, 0]$ and *increases* on $[0, \infty)$. It is *monotonic* on $(-\infty, 0]$ and on $[0, \infty)$ but *not monotonic* on $(-\infty, \infty)$.

First derivative test for monotonic functions

Theorem

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$. If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

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Example: Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

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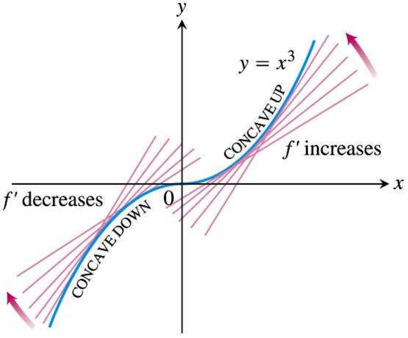
Example: Find the critical points of $f(x) = x^{4/3} - 4x^{1/3}$, identify the intervals on which f is increasing and decreasing, and find the function's extrema.

DEFINITION Concave Up, Concave Down

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I

(b) **concave down** on an open interval I if f' is decreasing on I.



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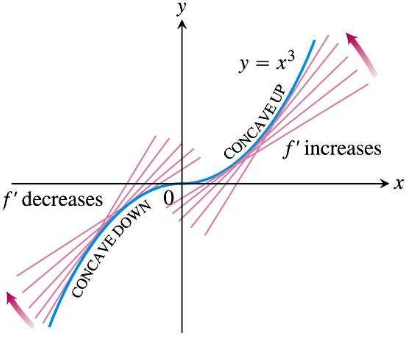

In the literature 'concave up' is often referred to as *convex*, and 'concave down' is simply called *concave*.

The second derivative test for concavity

If f is twice differential on an interval I, the First Derivative Test for Monotonic Functions implies that f' *increases* on I if $f''(x) > 0$ for all $x \in I$ and *decreases* if $f''(x) < 0$ for all $x \in I$. This gives:

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The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

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Example Find the intervals on the real line for which the graphs of the following functions are concave up or concave down:
(1) $y = x^3$
(2) $y = x^2$

DEFINITION Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

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Example: $y = x^{1/3}$.

Points of inflection

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Example $y = x^4$.