Queen Mary
University of London

MTH4100 Calculus I

Lecture notes for Week 8

Thomas' Calculus, Sections 4.1 to 4.4

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Extreme values of functions

DEFINITIONS Absolute Maximum, Absolute Minimum

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

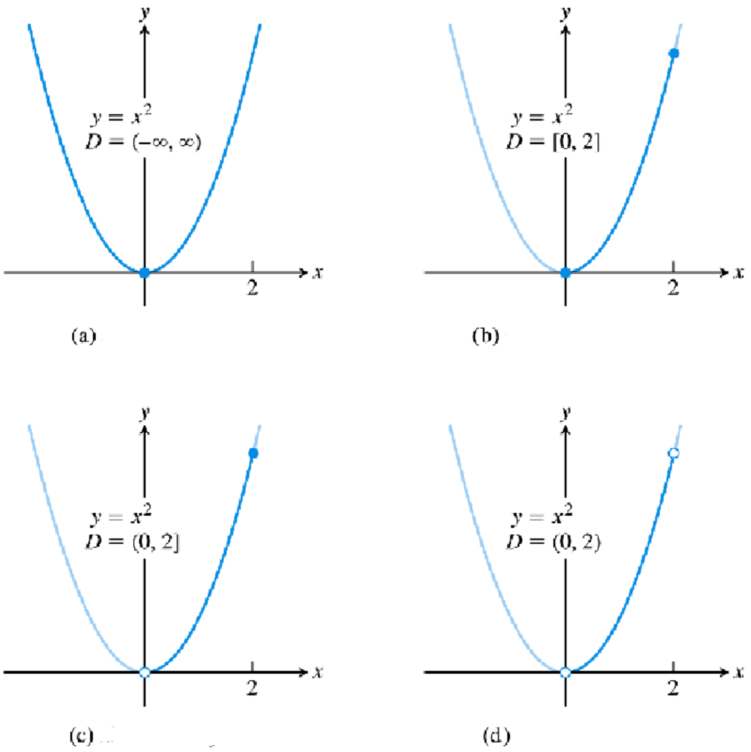
$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

These values are also called *absolute extrema*, or *global extrema*.

Example:

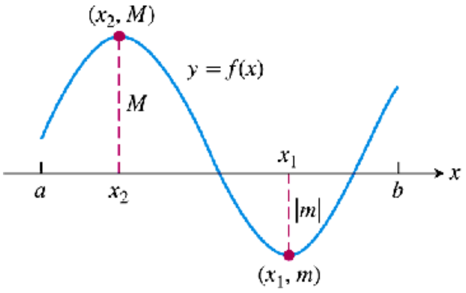


	Domain	abs. max.	abs. min.
(a)	$(-\infty, \infty)$	none	0, at $x = 0$
(b)	$[0, 2]$	4, at $x = 2$	0, at $x = 0$
(c)	$(0, 2]$	4, at $x = 2$	none
(d)	$(0, 2)$	none	none

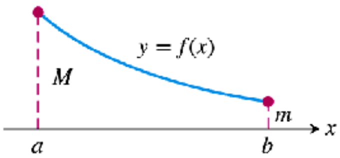
When the domain of f is a closed interval, the existence of a global maximum and minimum is ensured by:

THEOREM 1 The Extreme Value Theorem
If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$ (Figure 4.3).

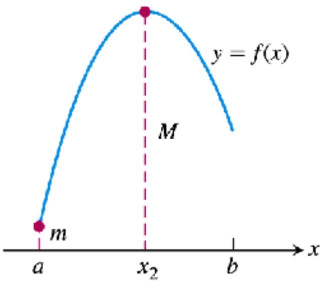
Examples:



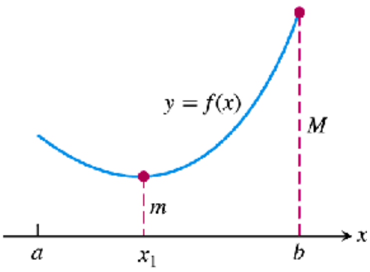
Maximum and minimum at interior points



Maximum and minimum at endpoints



Maximum at interior point, minimum at endpoint



Minimum at interior point, maximum at endpoint

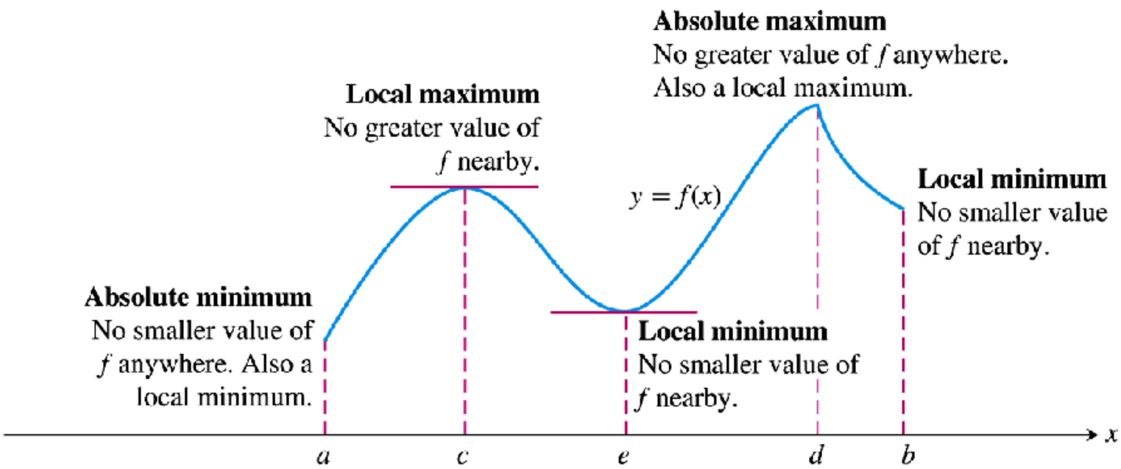
DEFINITIONS Local Maximum, Local Minimum
A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

These values are also called *local extrema*.



Note: Absolute extrema are automatically local extrema, but the converse need not be true. We can find local extreme points for a differential function by using the following result.

Theorem 1 (First Derivative Theorem for Local Extrema) *Suppose that f has a local maximum or minimum value at an interior point c of its domain, and that f is differentiable at c. Then $f'(c) = 0$.*

Proof idea: Lets suppose that $f(c)$ is a local maximum - the proof for a local minimum is similar. We have

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

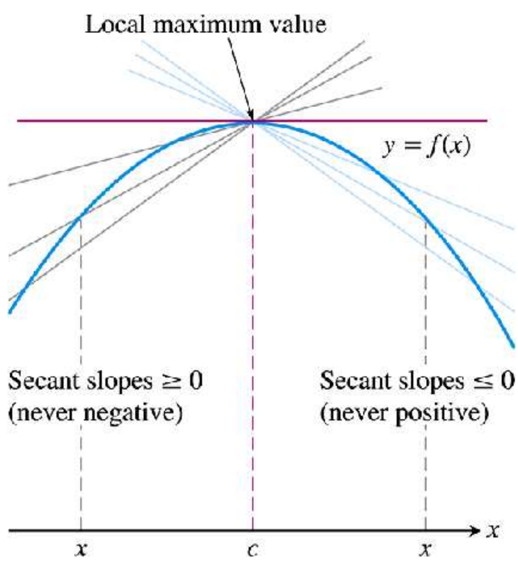
Consider the right and left hand limits separately. Since $f(c)$ is local maximum of c we have $f(x) - f(c) \leq 0$ for all x sufficiently close to c. Thus

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

and

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0.$$

Hence $f'(c) = 0$.



Note: the converse is false! *(counterexample?)*

Where can a function f possibly have an extreme value according to this theorem?

Answer:

1. at interior points where $f' = 0$
2. at interior points where f' is not defined
3. at endpoints of the domain of f.

Points at which 1 or 2 occur are combined in the following definition:

> **DEFINITION Critical Point**
> An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

The Extreme Value Theorem tells us that a continuous function f on a bounded closed interval has absolute maximum and minimum values. The First Derivative Theorem for Local Extrema gives us a method to determine these values:

Step 1 Determine the citical points of f.

Step 2 Evaluate f at each critical point AND at the end points of the interval.

Step 3 Take the largest and smallest values appearing in Step 2.

Examples: (1) Find the absolute extrema of $f(x) = x^2$ on $[-1, 1]$.

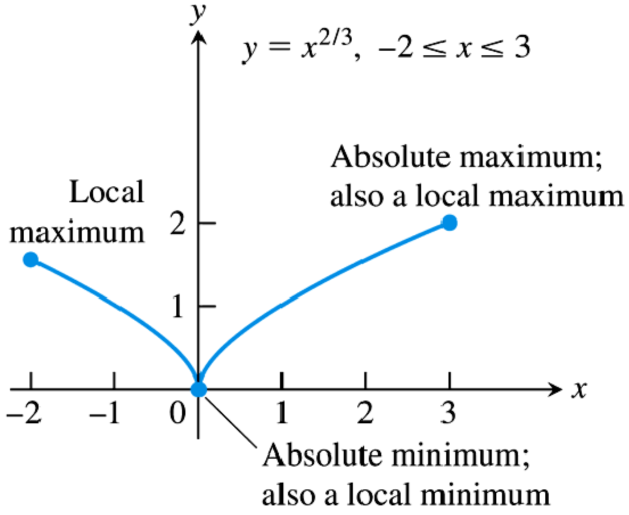
- f is differentiable on $[-1, 1]$ with $f'(x) = 2x$
- critical points: $f'(x) = 0 \quad \Rightarrow \quad x = 0$
- endpoints: $x = -1$ and $x = 1$
- Evaluate f at all critical points and endpoints: $f(0) = 0,\ f(-1) = 1,\ f(1) = 1$

Therefore f has an *absolute maximum value* of 1 which occurs at $x = \pm 1$, and an *absolute minimum value* of 0 which occurs at $x = 0$.

(2) Find the absolute extrema of $f(x) = x^{2/3}$ on $[-2, 3]$.

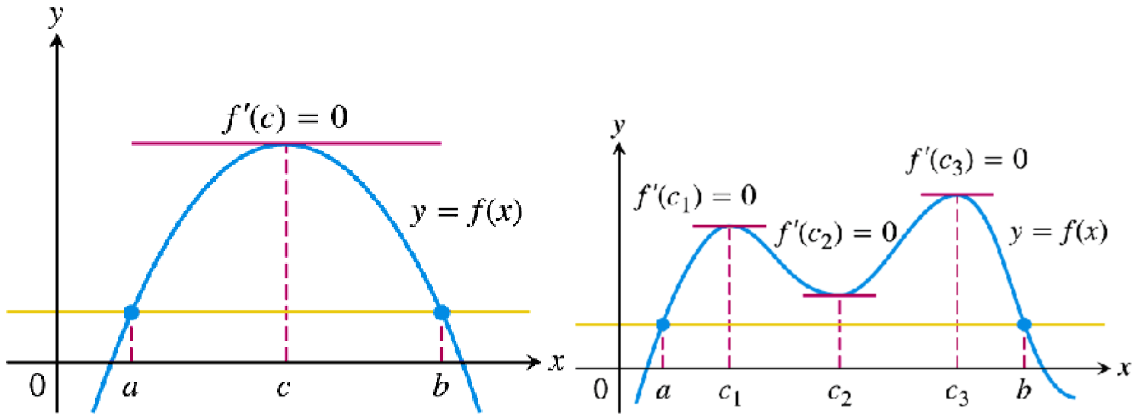
- f is differentiable with $f'(x) = \frac{2}{3}x^{-1/3}$ *except* at $x = 0$
- critical points: $f'(x) = 0$ or $f'(x)$ undefined $\Rightarrow \quad x = 0$
- endpoints: $x = -2$ and $x = 3$
- Evaluate f at all critical points and endpoints: $f(-2) = \sqrt[3]{4},\ f(0) = 0,\ f(3) = \sqrt[3]{9}$

Therefore f has an *absolute maximum value* of $\sqrt[3]{9}$ at $x = 3$ and an *absolute minimum value* of 0 at $x = 0$.



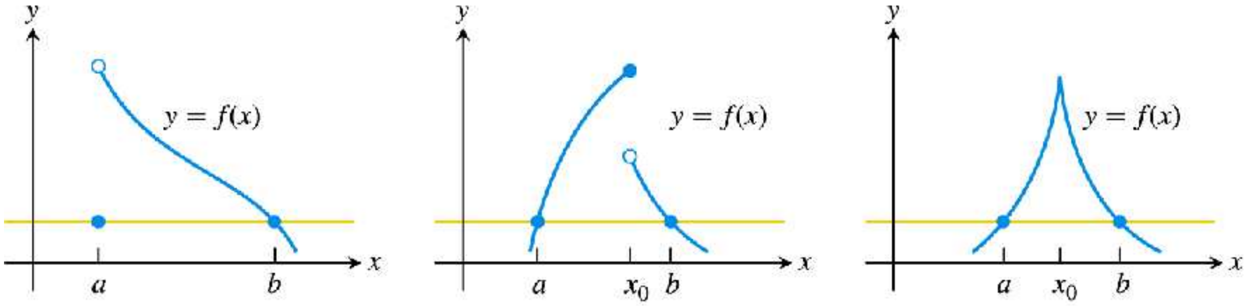
Rolle's theorem
This result tells us that a function which is continuous on a bounded closed interval and takes the same value at the both endpoints of the interval must have at least one critical point in the interval.

Theorem 2 (Rolle's Theorem) *Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$ then there exists a $c \in (a, b)$ with $f'(c) = 0$.*



Proof idea:
Apply extreme value theorem and first derivative theorem for extrema to interior points and consider endpoints separately; for details see the textbook Section 4.2.

Note: It is essential that *both* the hypotheses in the theorem are fulfilled i.e. f is continuous on $[a, b]$ and differentiable on (a, b):



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

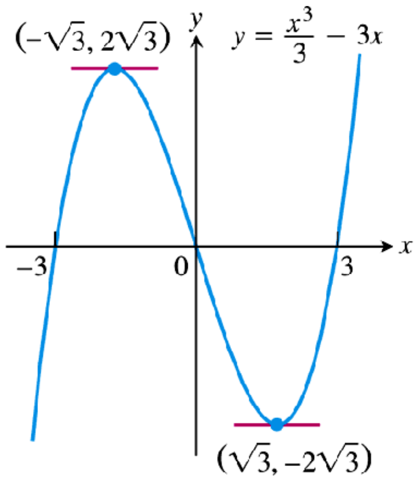
In each case there is no point $c \in (a, b)$ with $f'(c) = 0$.

Example: Apply Rolle's theorem to $f(x) = \frac{x^3}{3} - 3x$ on $[-3, 3]$.

- The function f is continuous on $[-3, 3]$, differentiable on $(-3, 3)$, and satisfies $f(-3) = f(3) = 0$.

- By Rolle's theorem there exists (at least) one $c \in [-3, 3]$ with $f'(c) = 0$.

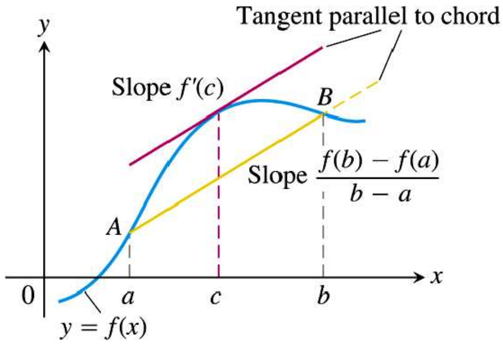
From $f'(x) = x^2 - 3 = 0$ we find that $f'(x) = 0$ when $x = \pm\sqrt{3}$.



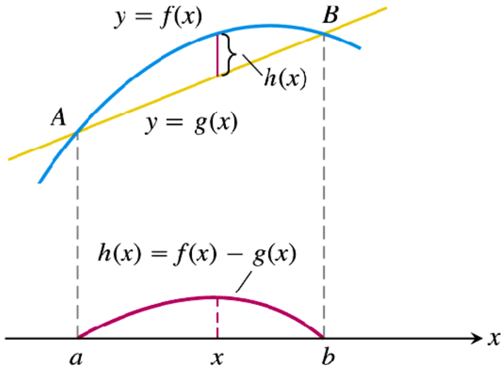
The Mean Value Theorem
This extends Rolle's theorem to the case when $f(a) \neq f(b)$.

Theorem 3 (Mean Value Theorem) *Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ with*

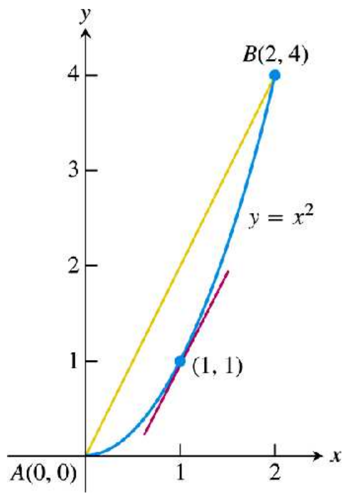
$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$



Proof idea: Let $y = g(x)$ be the equation of the line through the points $A = (a, f(a))$ and $B = (b, f(b))$. Define a new function h by putting $h(x) = f(x) - g(x)$. Then $h(a) = 0 = h(b)$ and we can deduce the Mean Value Theorem for f by applying Rolle's Theorem to h.



Example: Consider $f(x) = x^2$ on $[0, 2]$.



- $f(x)$ is continuous and differentiable on $[0, 2]$.
- Therefore there is a $c \in (0, 2)$ with $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = 2$.
- Since $f'(x) = 2x$, we can solve $f'(c) = 2c = 2$ to find that $c = 1$.

The following corollaries use the Mean Value Theorem to deduce information about a function from its derivative.

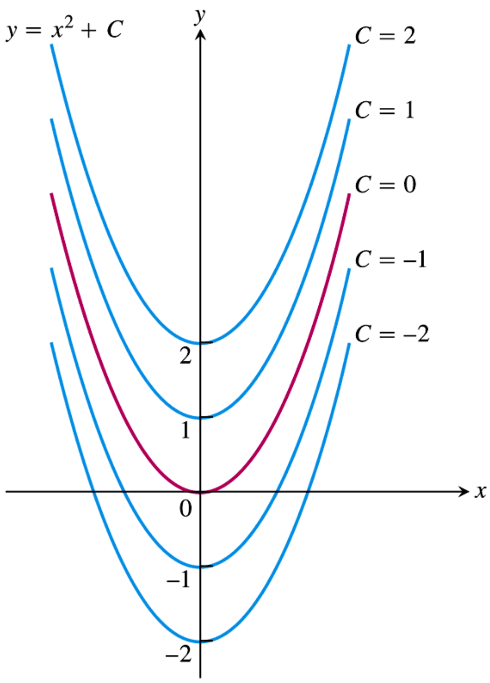
Corollary 1 (Functions with zero derivatives are constant) *If $f'(x) = 0$ for all $x \in (a, b)$ then $f(x_1) = f(x_2)$ for all $x_1, x_2 \in (a, b)$.*

Proof idea:
Apply the Mean Value Theorem to the closed interval $[x_1, x_2]$.

Corollary 2 (Functions with the same derivative differ by a constant) *If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + C$ for some constant $C \in \mathbb{R}$.*

Proof: Consider $h(x) = f(x) - g(x)$. As $h'(x) = f'(x) - g'(x) = 0$ for all $x \in (a, b)$, $h(x) = C$ for some constant $C \in \mathbb{R}$ by the previous corollary. Hence $f(x) = g(x) + C$. •

Example: If $f(x) = x^2$ then $f'(x) = 2x$. Hence every function h for which $h'(x) = f'(x) = 2x$ is of the form $h(x) = f(x) + C = x^2 + C$ for some $C \in \mathbb{R}$.



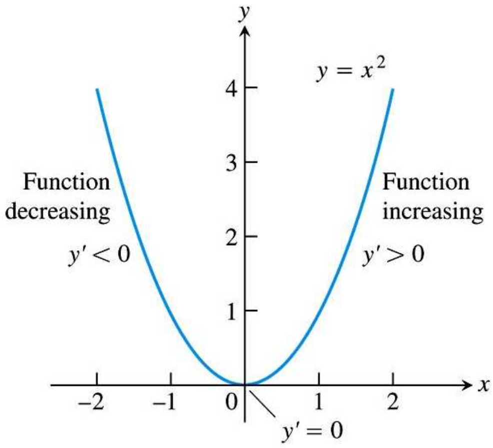
Monotonic functions

DEFINITIONS Increasing, Decreasing Function
Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

A function that is increasing or decreasing on I is called **monotonic** on I.

Example: $f(x) = x^2$ *decreases* on $(-\infty, 0]$ and *increases* on $[0, \infty)$. It is *monotonic* on $(-\infty, 0]$ and on $[0, \infty)$ but *not monotonic* on $(-\infty, \infty)$.



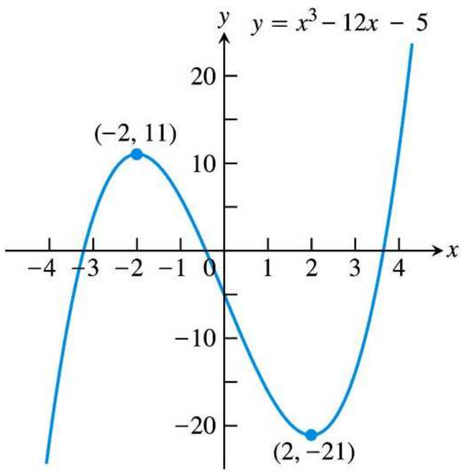
Theorem 4 (First derivative test for monotonic functions) *Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).*
If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Proof idea:
The Mean Value theorem tells us that, for any $x_1, x_2 \in [a, b]$ with $x_1 < x_2$, there exists a $c \in [x_1, x_2]$ with $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Hence, the sign of $f'(c)$ determines whether $f(x_2) < f(x_1)$ or $f(x_2) < f(x_1)$.

Example: Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing.

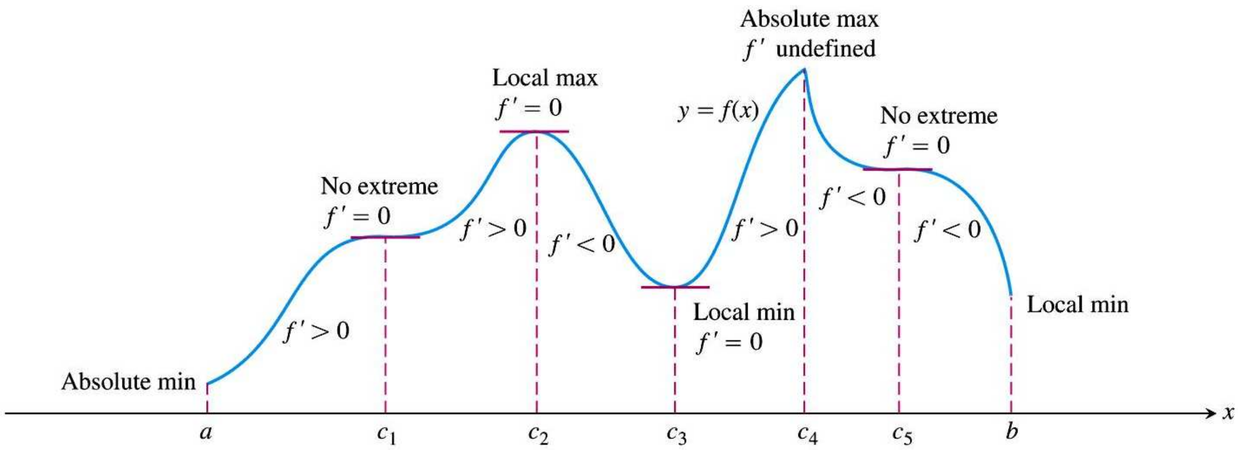
We have $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2)$. Hence $f'(x) = 0$ implies that $x = -2$ or $x = 2$. These critical points subdivide the natural domain of f, $(-\infty, \infty)$, into three subintervals $(-\infty, -2), (-2, 2), (2, \infty)$. Since f' is continuous it will have constant sign on each of these subintervals by the Intermediate Value Theorem. Hence we can determine the sign of f' on each subinterval by computing $f'(x)$ at *one* point x in the subinterval. We have: $f'(-3) = 15$, $f'(0) = -12$, $f'(3) = 15$. Thus

interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
sign of f'	$+$	$-$	$+$
behaviour of f	increasing	decreasing	increasing



First derivatives and local extrema

Example:



- Whenever f has a local minimum, $f' < 0$ to the left and $f' > 0$ to the right.
- Whenever f has a local maximum, $f' > 0$ to the left and $f' < 0$ to the right.

This implies that the sign of f' changes at local extrema.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

Example: Find the critical points of $f(x) = x^{4/3} - 4x^{1/3}$, identify the intervals on which f is increasing and decreasing, and find the function's extrema.
We have

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4(x-1)}{3x^{2/3}} = \frac{4(x-1)}{3(x^{1/3})^2}.$$

Hence f has two critical points at $x = 1$ and $x = 0$ and we have:

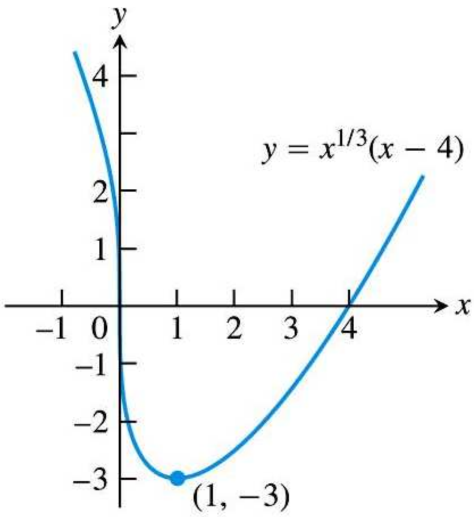
intervals	$x < 0$	$0 < x < 1$	$1 < x$
sign of f'	$-$	$-$	$+$
behaviour of f	decreasing	decreasing	increasing

Apply the first derivative test to identify local extrema:

- f' does not change sign at $x = 0$ so f has no extremum at $x = 0$;

- f' changes sign from $-$ to $+$ at $x = 1$ so f has a local minimum at $x = 0$.

Note that, since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$, the local minimum at $x = 1$ with $f(1) = -3$ is also an *absolute minimum.*
Note also that $\lim_{x\to 0} f'(x) = -\infty$ so the function is decreasing 'infinitely fast' when $x = 0$.

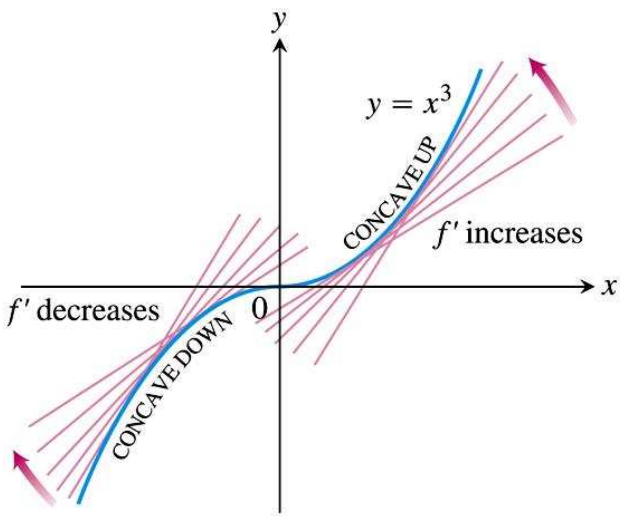


Concave Functions

DEFINITION Concave Up, Concave Down
The graph of a differentiable function $y = f(x)$ is

(a) concave up on an open interval I if f' is increasing on I

(b) concave down on an open interval I if f' is decreasing on I.



intervals	$x < 0$	$0 < x$
turning of curve	turns to the *right*	turns to the *left*
tangent slopes	decreasing	increasing

The turning or bending behaviour defines the *concavity* of the curve.
In the literature you often find that 'concave up' is referred to as *convex*, and 'concave down' is simply called *concave*.

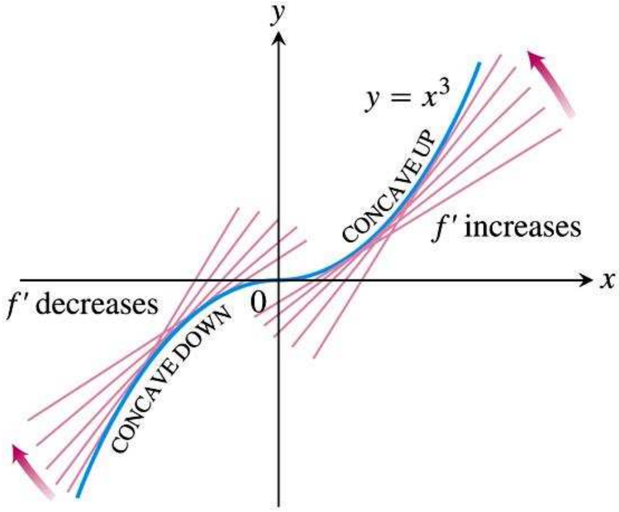
If f is twice differential on an interval I, the First Derivative Test for Monotonic Functions implies that f' *increases* on I if $f''(x) > 0$ for all $x \in I$ and *decreases* if $f''(x) < 0$ for all $x \in I$. This gives:

The Second Derivative Test for Concavity

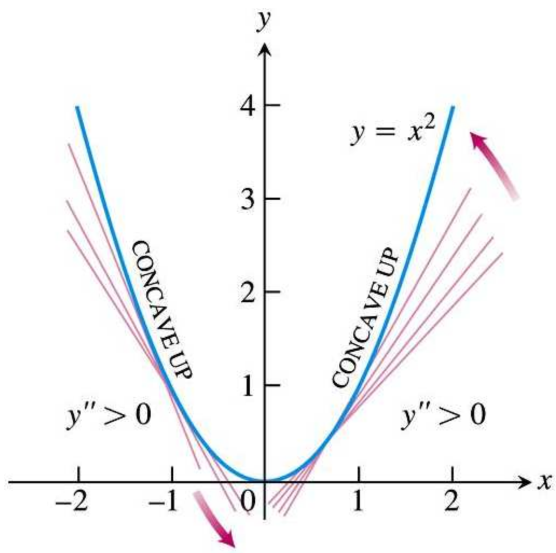
Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Examples: (1) $y = x^3$. We have $y'' = 6x$. For $x \in (-\infty, 0)$, $y''(x) < 0$ and so the graph is concave down. For $x \in (0, \infty)$, $y''(x) > 0$ and the graph is concave up.



(2) $y = x^2$. We have $y''(x) = 2 > 0$ for all $x \in \mathbb{R}$. Hence the graph is concave up everywhere.



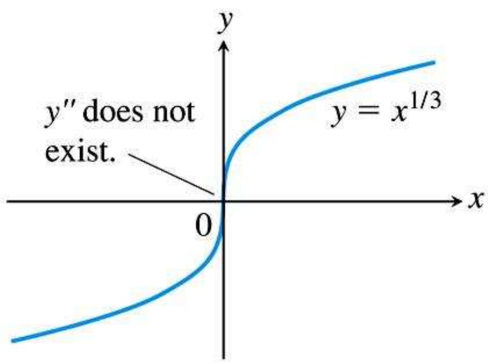
Points of inflection

We saw in the first example above that the graph of $y = x^3$ changes concavity at the point $(0, 0)$. Such a point is covered by the following definition.

DEFINITION **Point of Inflection**
A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

The condition that the graph of the function has a tangent line at a point is more general than saying that the function is differentiable at the point since it allows the tangent line to be vertical (and hence the derivative to be 'infinite').

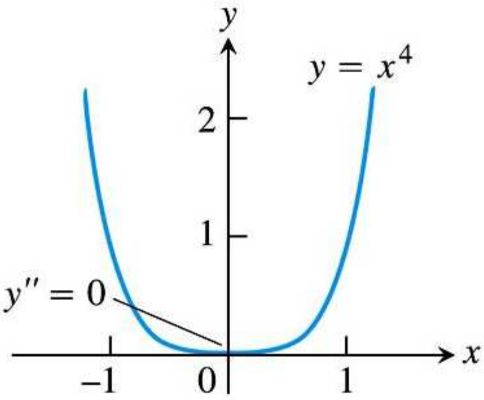
Example Consider $y = x^{1/3}$. We have $y' = \frac{1}{3}x^{-\frac{2}{3}}$ and $y'' = -\frac{2}{9}x^{-\frac{5}{3}}$. Hence y'' does not exist at $x = 0$. On the other hand $\lim_{x\to 0^-} y''(x) = \infty$ and $\lim_{x\to 0^+} y''(x) = -\infty$. Thus y'' changes sign as we pass through $x = 0$ and we do have a point of inflection at $x = 0$ (even though $y''(0)$ does not exist).



Suppose f is a function. At a point of inflection $(c, f(c))$ of f we have $f''(x) > 0$ on one side of c, $f''(x) < 0$ on the other side of c, and either $f''(c) = 0$ or f'' is undefined at c itself. Thus, if $f''(c)$ exists, then $(c, f(c))$ is a point of inflection if and only if $f''(c) = 0$ AND f' has a local maximum or minimum at $x = c$.
Example: Consider $f(x) = x^3 - 3x$. We have $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. Since $f''(0) = 0$ and $f'(0) = -3$ is a local minimum of f', the graph of f has a point of inflection at $x = 0$.

Note, however, that we can have $f''(c) = 0$ without $(c, f(c))$ being a point of inflection (when f' does not change sign at $x = c$).

Example Consider $y = x^4$. We have $y' = 4x^3$ and $y'' = 12x^2$. Thus $y''(0) = 0$. BUT y'' does not change sign at $x = 0$. Hence there is no inflection point at $x = 0$.


y
y = x^4
2
1
y'' = 0
x
−1
0
1