MTH4101 Calculus II

Lecture notes for Week 8
Series III and Integration III

Thomas' Calculus, Sections 10.8 to 10.10 and 15.1

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Taylor and Maclaurin Series

Assume that the function $f(x)$ can be represented as a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n + \cdots$$

and converges for $a - R < x < a + R$ with $R > 0$. Can we calculate the coefficients a_n in terms of $f(x)$?
It can be shown[1] that $f(x)$ has *derivatives of all orders* inside this interval by *differentiating the power series term by term*:

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x-a) + \cdots + na_n(x-a)^{n-1} + \cdots \\ f''(x) &= 1 \cdot 2a_2 + 2 \cdot 3a_3(x-a) + \cdots + n(n-1)a_n(x-a)^{n-2} + \cdots \\ &\vdots \\ f^{(n)}(x) &= n!\, a_n + \text{ a sum of terms with } (x-a) \text{ as a factor.} \end{aligned}$$

Therefore

$$f'(a) = a_1\,,\ f''(a) = 1 \cdot 2a_2\,,\ f'''(a) = 1 \cdot 2 \cdot 3a_3\,,\ \ldots\,, f^{(n)}(a) = n!\, a_n\,.$$

This gives us a formula for the coefficients in the power series:

$$a_n = \frac{f^{(n)}(a)}{n!}\,.$$

It also suggest that *if* f has a power series representation *then* it must be

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots\,.$$

leading us to the following definition:

DEFINITIONS **Taylor Series, Maclaurin Series**

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

[1] This is a theorem, which can be proved. Likewise, it can be proved that $f(x)$ can be *integrated term by term*; see Thomas' Calculus, end of Section 10.7. for details.

Example:
Find the Taylor series generated by $f(x) = 1/x$ at $x = 2$. Where, if anywhere, does the series converge to $1/x$?

$$\begin{aligned}
f(x) &= x^{-1}\,; \quad f(2) = 2^{-1} = \frac{1}{2} \\
f'(x) &= -x^{-2}\,; \quad f'(2) = -\frac{1}{2^2} \\
f''(x) &= 2!\,x^{-3}\,; \quad \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3} \\
&\vdots \\
f^{(n)}(x) &= (-1)^n n!\,x^{-(n+1)}\,; \quad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}\,.
\end{aligned}$$

The Taylor series is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots\,.$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$, or $0 < x < 4$ with sum

$$S = \frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x}\,.$$

Related to the Taylor *series* is the Taylor *polynomial* of order n:

DEFINITION **Taylor Polynomial of Order n**

Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

There is a similar definition for Maclaurin polynomials.

Example:
Find the Taylor polynomials of order 0, 2 and 4 for the function $f(x) = \cos x$ at $x = 0$.
We have

$$f(x) = \cos x\,, \quad f'(x) = -\sin x\,, \quad f''(x) = -\cos x\,, \quad f'''(x) = \sin x\,, \quad f^{(4)}(x) = \cos x$$

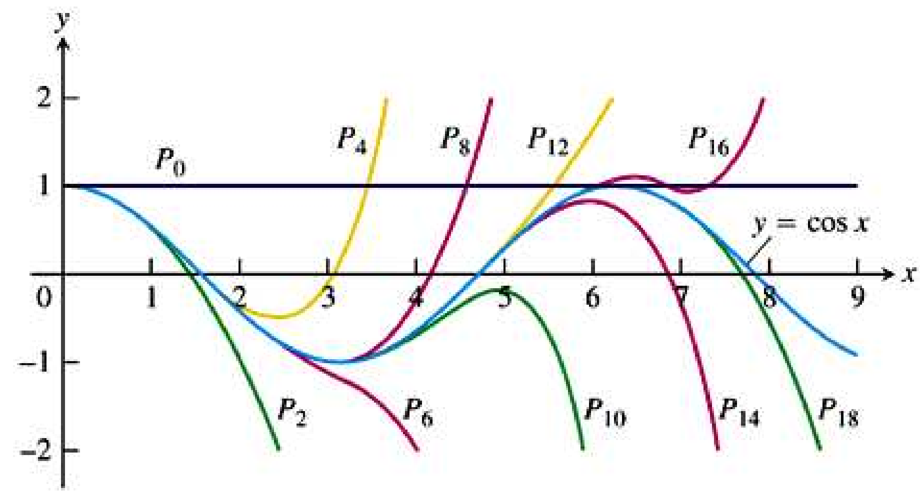
and

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^{(4)}(0) = 1\,.$$

By using the previous definition, the first three Taylor polynomials of $f(x) = \cos x$ about $x = 0$ are

$$\begin{aligned} P_0(x) &= 1 \\ P_2(x) &= 1 - \frac{x^2}{2!} \\ P_4(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}. \end{aligned}$$

The following figure shows how successive Taylor polynomials provide better and better approximations to the function as $n \to \infty$:



Below we give the Taylor series expansions for a variety of functions about $x = 0$ and $x = 1$. These can all be derived using the methods in this section.
Taylor series about $x = 0$:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \\ \sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots. \end{aligned}$$

Taylor series about $x = 1$:

$$\begin{aligned} \ln x &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots \\ \sqrt{x} &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \cdots. \end{aligned}$$

Convergence of Taylor Series and Error Estimates

There are still two unanswered questions about Taylor series:

1. **When** does a Taylor series **converge** to the function that generated it?
2. **How accurately** do a function's Taylor polynomials **approximate** the function on a given interval?

To answer these questions we need to make use of **Taylor's formula**:

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \tag{1}$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \tag{2}$$

This theorem can be understood as a generalization of the Mean Value Theorem (set $n = 0$ in the above formula).
The quantity $R_n(x)$ in Taylor's Formula is called the **remainder of order** n or the **error term** for the approximation of f by $P_n(x)$ over I. If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series *converges* to f on I and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k .$$

Finally we can use the **Remainder Estimation Theorem** to provide an estimate of the error:

THEOREM 23 The Remainder Estimation Theorem

If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}.$$

If this condition holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

The usefulness of this theorem is demonstrated by the following example:

Example:

Show that the Taylor series for $\sin x$ at $x = 0$ converges for all x.

We have

$$f(x) = \sin x\,, \quad f'(x) = \cos x\,, \quad f''(x) = -\sin x\,, \ldots$$

and, in general,

$$f^{(2k)}(x) = (-1)^k \sin x\,, \quad f^{(2k+1)}(x) = (-1)^k \cos x$$

Therefore, evaluating at $x = 0$ gives $f^{(2k)}(0) = 0$ and $f^{(2k+1)}(0) = (-1)^k$. Hence the Taylor series for $\sin x$ at $x = 0$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x)\,.$$

Applying the Remainder Estimation Theorem with $M = 1$ gives

$$|R_{2k+1}(x)| \leq 1 \cdot \frac{|x|^{2k+2}}{(2k+2)!} \to 0 \text{ as } k \to \infty \text{ for all } x.$$

(cf. the list of sequences and their limits discussed in Week 5) Therefore $R_{2k+1}(x) \to 0$ and the Maclaurin series for $\sin x$ converges to $\sin x$ for every x.

Applications of Power Series

Binomial series

The Taylor series generated by $f(x) = (1+x)^m$ (around $x = 0$) where m is a constant is

$$\begin{aligned} f(x) &= 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots \\ &\quad + \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!}x^k + \cdots\,. \end{aligned}$$

This is called the **binomial series**.

If $m \geq 0$ is an integer, the series stops after $(m+1)$ terms because coefficients from $k = m+1$ onwards are zero.

If m is not a positive integer the series is *infinite*. From the Ratio Test for absolute convergence it follows that this series converges absolutely for $|x| < 1$. It can also be shown that the series converges to $(1+x)^m$.

We can define this series conveniently as follows:

The Binomial Series

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k \geq 3.$$

Note that $m \in \mathbb{R}$. In the case of $m \in \mathbb{N}$ we recover the familiar *binomial coefficients*. Note also the relation between the binomial series and the binomial formula.
In the case where $m = -1$,

$$\binom{-1}{1} = -1\,, \quad \binom{-1}{2} = 1 \quad \text{and} \quad \binom{-1}{k} = (-1)^k\,.$$

For example,

$$\begin{aligned} \frac{x}{1+x^2} &= x(1+x^2)^{-1} \\ &= x\left(1 - x^2 + \frac{(-1)(-2)}{2!}x^4 + \frac{(-1)(-2)(-3)}{3!}x^6 + \cdots\right) \\ &= x(1 - x^2 + x^4 - x^6 + \cdots) \\ &= x - x^3 + x^5 - x^7 + \cdots \end{aligned}$$

which is a geometric series.

Reading assignment: Work yourself through the following two examples. (cf. Examples 3 and 7 in Thomas' Calculus, Section 10.10)

Evaluation of non-elementary integrals

We can use the term-by-term integration property of power series to allow us to do non-elementary integrals.

Example:
Express $\int \sin x^2 \, \mathrm{d}x$ as a power series.
Recall that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Hence

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

and so

$$\int \sin x^2 \, \mathrm{d}x = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \cdots .$$

where C is a constant of integration.

Evaluating indeterminate forms

Power series also provide an alternative to L'Hôpital's rule for evaluating indeterminate forms.

Example:
Find

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) .$$

We can write

$$\begin{aligned} \frac{1}{\sin x} - \frac{1}{x} &= \frac{x - \sin x}{x \sin x} = \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)}{x \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)} \\ &= \frac{x^3 \left(\frac{1}{3!} - \frac{x^2}{5!} + \cdots \right)}{x^2 \left(1 - \frac{x^2}{3!} + \cdots \right)} = x \frac{\left(\frac{1}{3!} - \frac{x^2}{5!} + \cdots \right)}{\left(1 - \frac{x^2}{3!} + \cdots \right)} . \end{aligned}$$

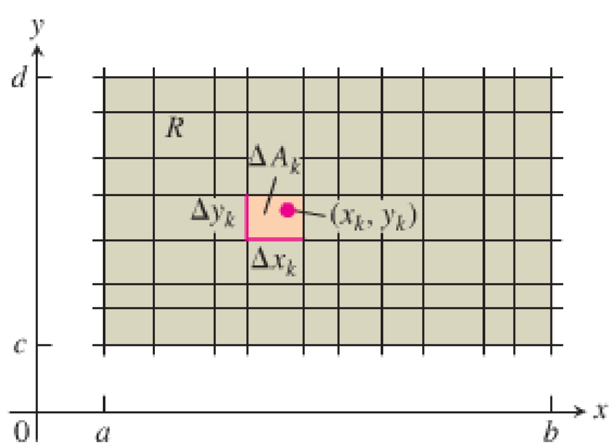
Hence

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \left(x \frac{\left(\frac{1}{3!} - \frac{x^2}{5!} + \cdots \right)}{\left(1 - \frac{x^2}{3!} + \cdots \right)} \right) = 0 .$$

Note that since $1/\sin x - 1/x \approx x \cdot (1/3!) = x/6$, we can write $\operatorname{cosec} x \approx (1/x) + (x/6)$.

Double Integrals

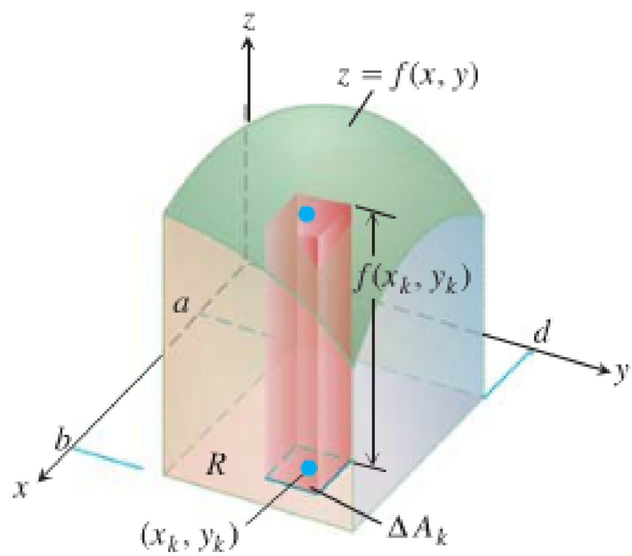
Consider a function $f(x, y)$ defined on a rectangular region $R : \; a \le x \le b, \; c \le y \le d$ partitioned into small rectangles A_k:



The area of a small rectangle with sides Δx_k and Δy_k is

$$\Delta A_k = \Delta x_k \, \Delta y_k \, .$$

Choose a point (x_k, y_k) in the (suitably numbered) kth rectangle with function value $f(x_k, y_k)$. We can consider $z = f(x, y)$ as defining the height z at the point (x, y). The product $f(x_k, y_k)\,\Delta A_k$ is then the *volume of a solid* with base area ΔA_k and height $f(x_k, y_k)$ (for which we assume that $f(x_k, y_k) > 0$):



The **Riemann sum** S_n of these solids over R is

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k \, .$$

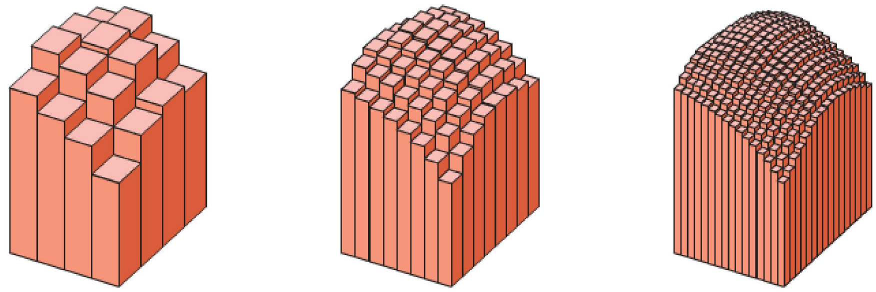
Now consider what happens as $\Delta A_k \to 0$ (as $n \to \infty$), i.e., we refine the partitioning. When the limit of these sums exists the function f is said to be **integrable** and the limit is called the **double integral** of f over R, written as

$$\int\!\!\int_R f(x, y)\,\mathrm{d}A \qquad \text{or} \qquad \int\!\!\int_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

The volume of the portion of the solid directly above the base ΔA_k is $f(x_k, y_k)\,\Delta A_k$. Hence the total volume above the region R is

$$\text{Volume} \; = \lim_{n\to\infty} S_n = \int\!\!\int_R f(x, y)\,\mathrm{d}A$$

where $\Delta A_k \to 0$ as $n \to \infty$. The following figure shows how the Riemann sum approximations of the volume become more accurate as the number n of boxes increases:



(a) $n = 16$ (b) $n = 64$ (c) $n = 256$