MTH4100 Calculus I

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Derivatives of trigonometric functions

$$\begin{aligned}
\frac{d}{dx}\sin x &= \cos x \\
\frac{d}{dx}\cos x &= -\sin x \\
\frac{d}{dx}\tan x &= \frac{1}{\cos^2 x} = \sec^2 x \\
\frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \sec x \tan x \\
\frac{d}{dx}\cot x &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = -\csc^2 x \\
\frac{d}{dx}\csc x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\csc x \cot x
\end{aligned}$$

Derivatives of compositions of functions

THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

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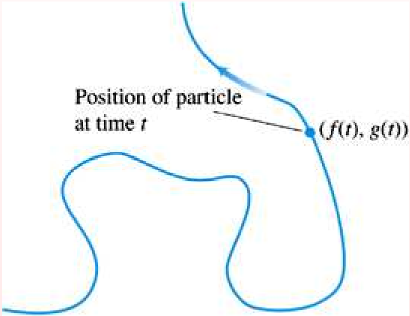
Example Differentiate $y = \sin(x^2 + x)$.

Parametric Curves

We can describe a point P moving in the xy-plane as a function of a *parameter* t ("time") by two functions $x = f(t)$ and $y = g(t)$ which give the coordinates of P at time t.



DEFINITION Parametric Curve

If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

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The variable t is the *parameter* for the curve. If the interval of possible t-values is $[a, b]$, then $[a, b]$ is called the *parameter interval*, the point $(f(a), g(a))$ is the *initial point* of the curve, and the point $(f(b), g(b))$ is the *terminal point* of the curve. The parametric equations and the parameter interval together form a *parametrisation* of the curve.

Parametric Curves - Example

Determine the curve defined by the parametrisation
$x = \sqrt{t}\ ,\ y = t\ ,\ t \in [0, \infty)$.

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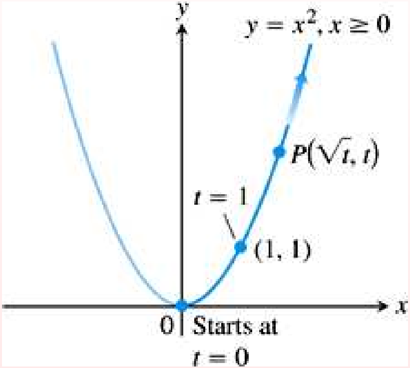

Find a parametrisation for the line segment in the xy-plane which joins the points $(-2, 1)$ and $(3, 5)$.

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It can be shown that if f and g are both differentiable at t then y is a differentiable function of x when $x = g(t)$. We can now use the chain rule to deduce that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \, .$$

Differentiable Curves

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Solving for dy/dx gives us a formula for the slope of the parametrised curve $x = f(t)$, $y = g(t)$ when it is differentiable at t and $dx/dt \neq 0$:

Parametric formula for dy/dx

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}\,.$$

Differentiable Curves - Example

Describe the motion of a particle whose position (x, y) at time t is given by

$$\boxed{x = a\cos t\ , \quad y = b\sin t\ , \quad 0 \leq t \leq 2\pi}$$

and compute the slope of this curve at time t.