Queen Mary
University of London

MTH4100 Calculus I

Lecture notes for Week 6

Thomas' Calculus, Sections 3.5 to 3.7, 3.9, 4.1, 11.1 (p.610-613) and 11.2 (p.618-619)

Prof Bill Jackson

School of Mathematical Sciences
Queen Mary University of London

Autumn 2012

Derivatives of trigonometric functions

(1) Differentiate $f(x) = \sin x$:

- Start with the **definition** of $f'(x)$:

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

- Use $\sin(x+h) = \sin x \cos h + \cos x \sin h$:

$$f'(x) = \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

- Collect terms and apply limit laws:

$$f'(x) = \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

- Use $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$ and $\lim_{h \to 0} \frac{\sin h}{h} = 1$ to conclude $f'(x) = \cos x$.

(2) A similar argument gives $\frac{d}{dx} \cos x = -\sin x$.

(3) We can now use the quotient rule to find the derivative of $\tan x$.

$$\begin{aligned}
\frac{d}{dx} \tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\frac{d}{dx}(\sin x)\cos x - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x}
\end{aligned}$$

Summary: Derivatives of trigonometric functions

$$\begin{aligned}
\frac{d}{dx} \sin x &= \cos x \\
\frac{d}{dx} \cos x &= -\sin x \\
\frac{d}{dx} \tan x &= \frac{1}{\cos^2 x} = \sec^2 x \\
\frac{d}{dx} \sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \sec x \tan x \\
\frac{d}{dx} \cot x &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = -\csc^2 x \\
\frac{d}{dx} \csc x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\csc x \cot x
\end{aligned}$$

Differentiating the composition of two functions

THEOREM 3 The Chain Rule

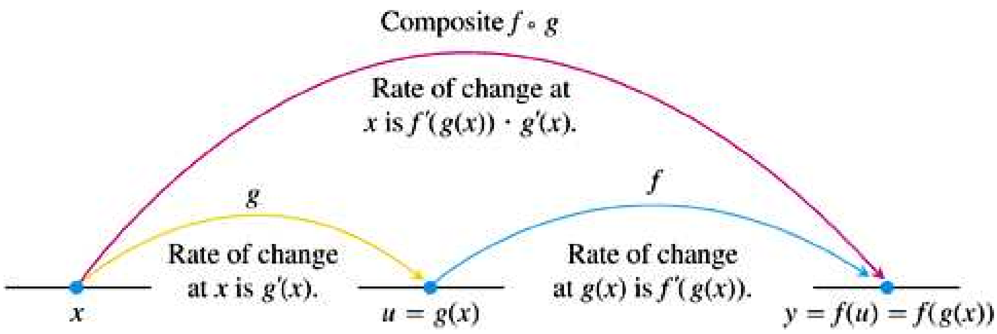
If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

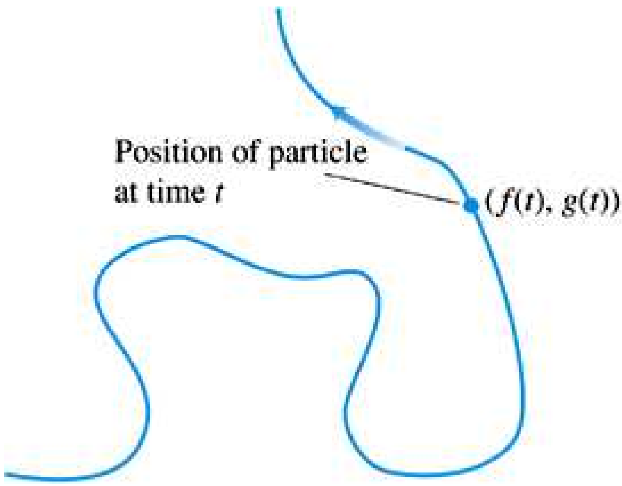


The chain rule tells us that the rate of change of $f \circ g$ at x is equal to the rate of change of g at x multiplied by the rate of change of f at $g(x)$.
Example: Differentiate $y = \sin(x^2 + x)$.
Let $u = x^2 + x$ and $y = \sin u$. Then $\frac{du}{dx} = 2x + 1$ and $\frac{dy}{du} = \cos u$. Hence

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (2x + 1)\cos(x^2 + x).$$

Parametric Curves

We can describe a point P moving in the xy-plane as a function of a *parameter* t ("time") by two functions $x = f(t)$ and $y = g(t)$ which give the coordinates of P at time t.



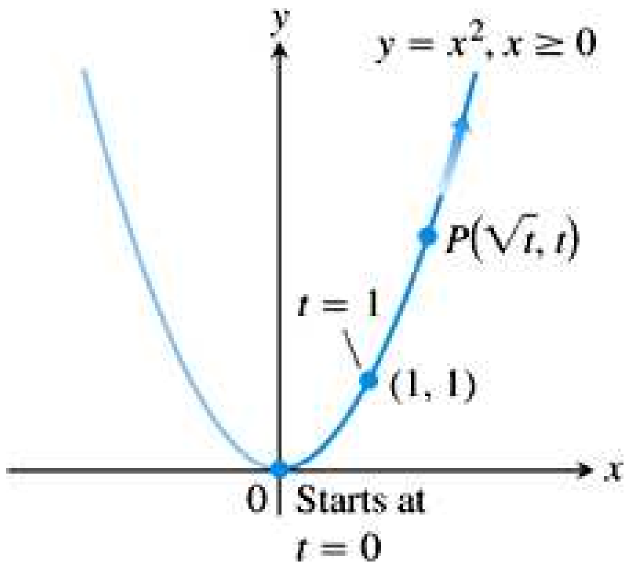
DEFINITION Parametric Curve
If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

The variable t is the *parameter* for the curve. If the interval of possible t-values is $[a, b]$, then $[a, b]$ is called the *parameter interval*, the point $(f(a), g(a))$ is the *initial point* of the curve, and the point $(f(b), g(b))$ is the *terminal point* of the curve. The parametric equations and the parameter interval together form a *parametrisation* of the curve.

Examples:
(1) Determine the curve defined by the parametrisation $x = \sqrt{t}\ ,\ y = t\ ,\ t \in [0, \infty)$.
In this example it is easy to solve the parametric equations and express y as a function of x: we have $y = t$ and $t = x^2$ so $y = x^2$. Note however that since $x = \sqrt{t}$, x only takes nonnegative values. Thus the curve is the segment of the parabola $y = x^2$ which lies in the positive quadrant.



(2) Find a parametrisation for the line segment in the xy-plane which joins the points $(-2, 1)$ and $(3, 5)$.
Let's suppose a point $P = (x(t), y(t))$ moves along the line segment starting at $(-2, 1)$ when $t = 0$ and ending at $(3, 5)$ when $t = 1$. Assuming the point moves at constant speed, its position at time t will be $(-2, 1) + t[(3, 5) - (-2, 1)] = (-2 + 5t, 1 + 4t)$. This gives the parametrisation: $x = -2 + 5t$ and $y = 1 + 4t$ for $t \in [0, 1]$.

Definition A parametrised curve $x = f(t)$, $y = g(t)$ is *differentiable* at t if f and g are both differentiable at t.

It can be shown that if f and g are both differentiable at t then y is a differentiable function of x when $x = g(t)$. We can now use the chain rule to deduce that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \,.$$

Solving for dy/dx gives us the following formula for the slope of the parametrised curve $x = f(t)$, $y = g(t)$ when it is differentiable at t and $dx/dt \neq 0$.

Parametric formula for dy/dx

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \,.$$

Example: Describe the motion of a particle whose position (x, y) at time t is given by

$$\boxed{x = a\cos t\,, \quad y = b\sin t\,, \quad 0 \leq t \leq 2\pi}$$

and compute the slope of this curve at time t.

- We first use the two parametric equations to eliminate t and find one equation involving only x and y. Using $\cos t = x/a$, $\sin t = y/b$ and $\cos^2 t + \sin^2 t = 1$ we obtain

  $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \,,$$

  which is the equation of an **ellipse**.

- We have $\frac{dx}{dt} = -a\sin t$ and $\frac{dy}{dt} = b\cos t$. The parametric formula for dy/dx now yields

  $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b\cos t}{-a\sin t} = -\frac{b^2 x}{a^2 y} \,.$$

  Thus the slope of the ellipse at the point (x, y) is $-(b^2 x)/(a^2 y)$.

Implicit differentiation

Suppose we have a curve consisting of all points in the xy-plane which satisfy an *implicit relation* between x and y, i.e. an equation of the form $F(x, y) = 0$, and we want to find its slope dy/dx. If we can solve the implicit relation $F(x, y) = 0$ for y to obtain an *explicit relation* $y = f(x)$ for some function f then we can just differentiate $f(x)$. We use implicit differentiation when it is not obvious how to solve $F(x, y) = 0$ for y.

Example: Given the functional relation $y^2 = x$, find dy/dx.

New method by differentiating *implicitly*:

- Differente *both sides* of the equation $y^2 = x$ with respect to x. Assuming y is a differentiable function of x we can use the chain rule to obtain

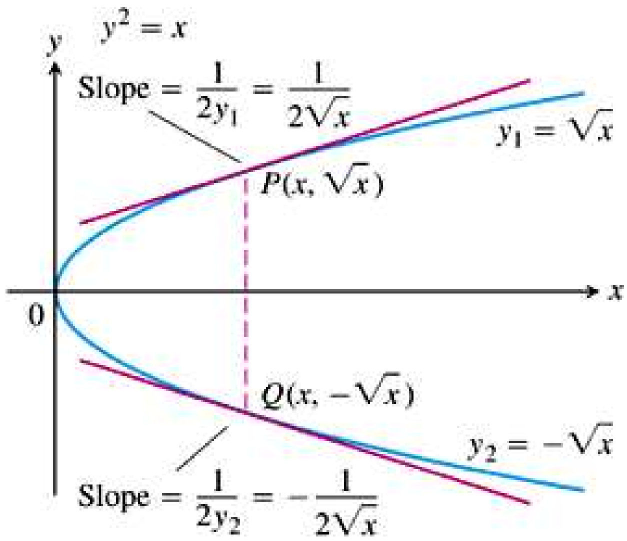
  $$2y\frac{dy}{dx} = 1.$$

- Solving for dy/dx we get
$$\frac{dy}{dx} = \frac{1}{2y}.$$

Compare with differentiating *explicitly*:

- We can solve $y^2 = x$ to obtain two *explicit solutions* for y: $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$. Thus the curve $y^2 = x$ is the union of the graphs of the two functions y_1 and y_2. The derivatives of these functions are:
$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \text{ and } \frac{dy_1}{dx} = -\frac{1}{2\sqrt{x}}$$

- We should compare this with the solution obtained by implicit differentiation. Substituting $y = y_1 = \sqrt{x}$ when $y > 0$ gives $\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$. Similarly substituting $y = y_2 = -\sqrt{x}$ when $y < 0$ gives $\frac{dy}{dx} = \frac{1}{2y} = -\frac{1}{2\sqrt{x}}$. Thus both solutions give the same value for dy/dx.



Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

Example: Use implicit differentiation to find dy/dx for the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
The three steps in the above method for implicit differentiation give:

1. $\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$

2. $\dfrac{2yy'}{b^2} = -\dfrac{2x}{a^2}$

3. $y' = -\dfrac{b^2}{a^2}\dfrac{x}{y}$.

This agrees with the result obtained previously using a parametrisation of the elipse.

Application: We can use implicit differentiation to calculate the derivative of the power function $y = x^a$ when a is a rational number, say $a = p/q$ for some integers p, q with $q \neq 0$:

- we have $$y^q = x^p$$
- implicit differentiation gives: $$qy^{q-1}\frac{dy}{dx} = px^{p-1}$$
- solving for $\frac{dy}{dx}$ as a function of x we obtain:

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\frac{x^p}{y^q}\frac{y}{x} = \frac{p}{q}\frac{y}{x} = \frac{p}{q}\frac{x^{\frac{p}{q}}}{x} = \frac{p}{q}x^{\frac{p}{q}-1}$$

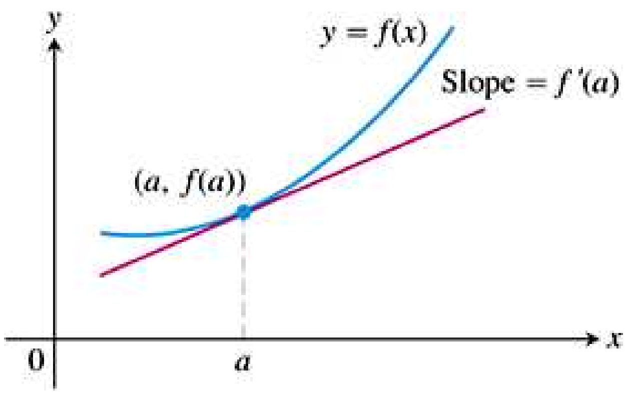
THEOREM 4 Power Rule for Rational Powers

If p/q is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$, and

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

Linearisation

We can use linearisation to replace a complicated function by a much simpler linear function if we are only interested in the values of the function close to a given point.



"Close to" the point $(a, f(a))$, the tangent $L(x) = f(a) + f'(a)(x - a)$ is a "good" approximation for $y = f(x)$.

DEFINITIONS Linearization, Standard Linear Approximation

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

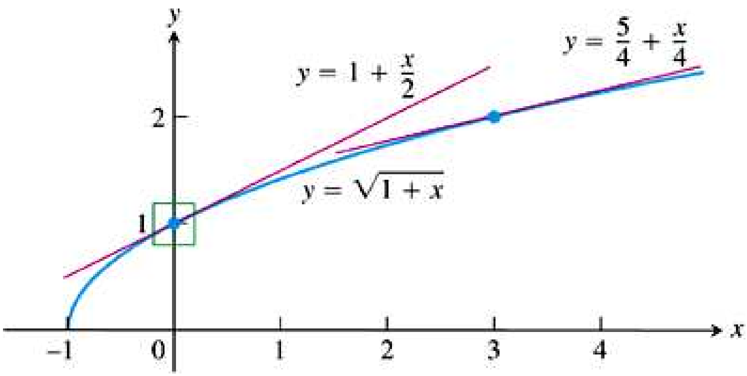
$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

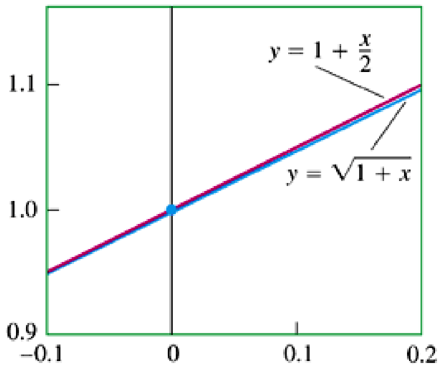
Example: Compute the linearisation of $f(x) = \sqrt{1+x}$ at $x = 0$.

We have $f(0) = 1$ and $f'(x) = \frac{1}{2}(1+x)^{-1/2}$. This gives $f'(0) = \frac{1}{2}$, so

$$L(x) = 1 + \frac{1}{2}x \ .$$



How accurate is this approximation? Magnify region around $x = 0$:



Approximation	True value	$\lvert$True value − approximation$\rvert$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$<10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$<10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$<10^{-5}$

Linearisations are used to simplify problems. For example if we are working on a problem which involves the values taken by $f(x) = \sqrt{1+x}$ on some small interval I centered on $x = 0$, then we can simplify our calculations and obtain an approximate solution by replacing $f(x)$ by $L(x) = 1 + \frac{x}{2}$ for all $x \in I$.

Differentials

The difference between the true value of a function $y = f(x)$ close to a point and its linearization can be made more precise using 'differentials'. When we write $y = f(x)$ we are thinking of x as an independent variable and y as a dependent variable. We introduce two new variable: dx, which is an independent variable measuring the distance we move from x; dy which is a dependent variable measuring the resultant change in the linearisation of $y = f(x)$ (and hence depends on both x and dx). The two new variables are called *differentials*. The dependency of y on x and dx is given by the equation for the linearisation of $f(x)$ centered at x: $L(x + dx) = L(x) + f'(x)([x + dx] - dx)$. Since $L(x) = f(x)$ and $L(x + dx) - L(x) = dy$ this gives:

$$dy = f'(x)dx$$

Reading Assignment: read
Thomas' Calculus, p. 167-168 about **Differentials**

Extreme values of functions

DEFINITIONS **Absolute Maximum, Absolute Minimum**
Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

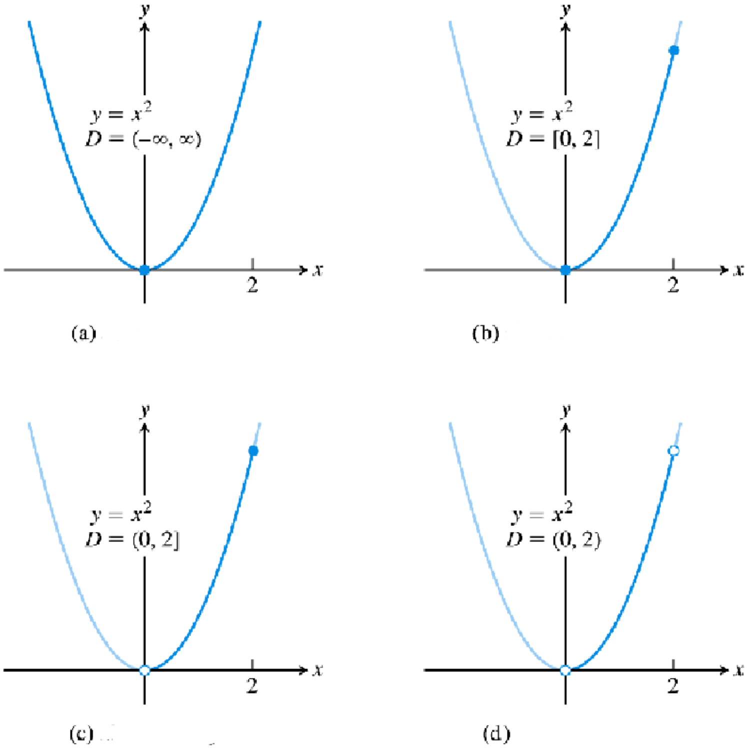
$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

These values are also called *absolute extrema*, or *global extrema*.

Example:



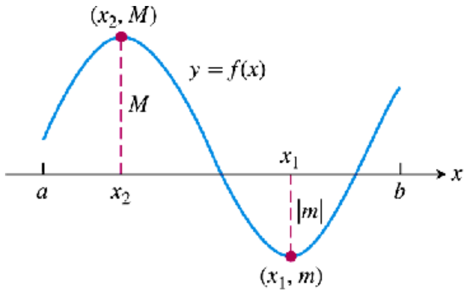
	Domain	abs. max.	abs. min.
(a)	$(-\infty, \infty)$	none	0, at 0
(b)	$[0, 2]$	4, at 2	0, at 0
(c)	$(0, 2]$	4, at 2	none
(d)	$(0, 2)$	none	none

When the domain of f is a closed interval, the existence of a global maximum and minimum is ensured by:

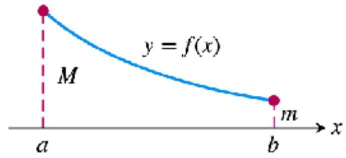
THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$ (Figure 4.3).

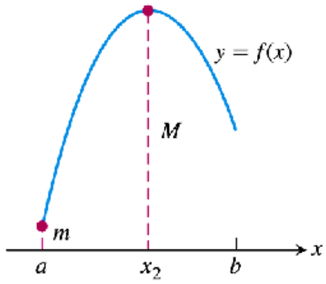
Examples:



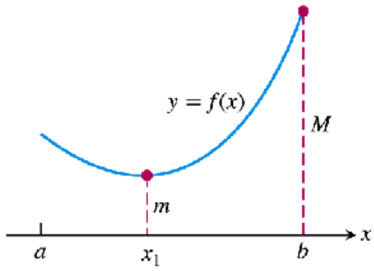
Maximum and minimum at interior points



Maximum and minimum at endpoints



Maximum at interior point, minimum at endpoint



Minimum at interior point, maximum at endpoint

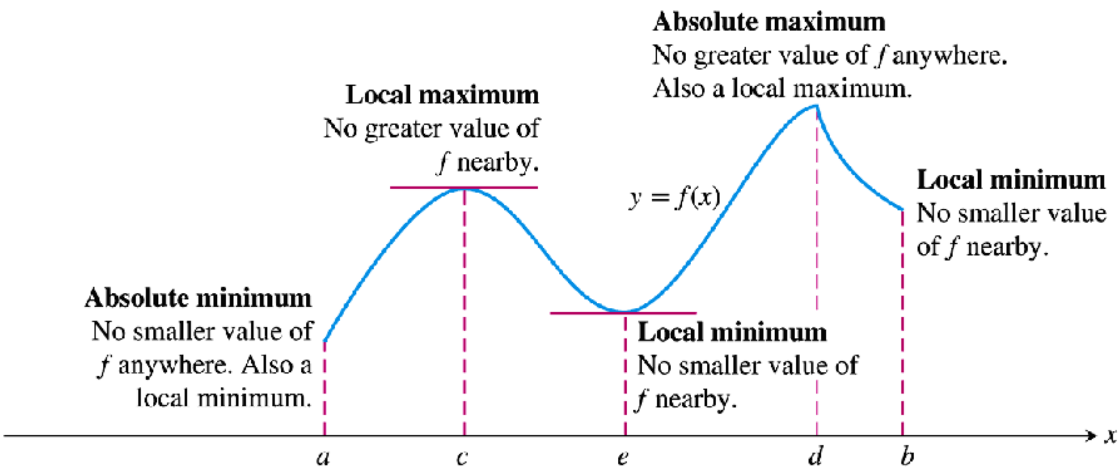
DEFINITIONS **Local Maximum, Local Minimum**

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$



Note: Absolute extrema are automatically local extrema, but the converse need not be true.