MTH4101 Calculus II

Lecture notes for Week 6
Series II and III

Thomas' Calculus, Sections 10.3 and 10.5 to 10.7

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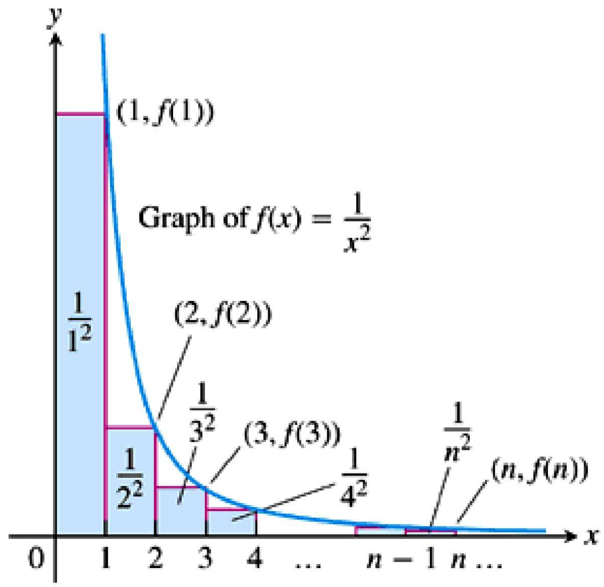


$$
\begin{aligned}
s_n &= \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2} \\
&= f(1)+f(2)+f(3)\cdots+f(n) \\
&< f(1)+\int_1^n \frac{1}{x^2}\,\mathrm{d}x \quad \text{lower sum} \\
&< 1+\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x
\end{aligned}
$$

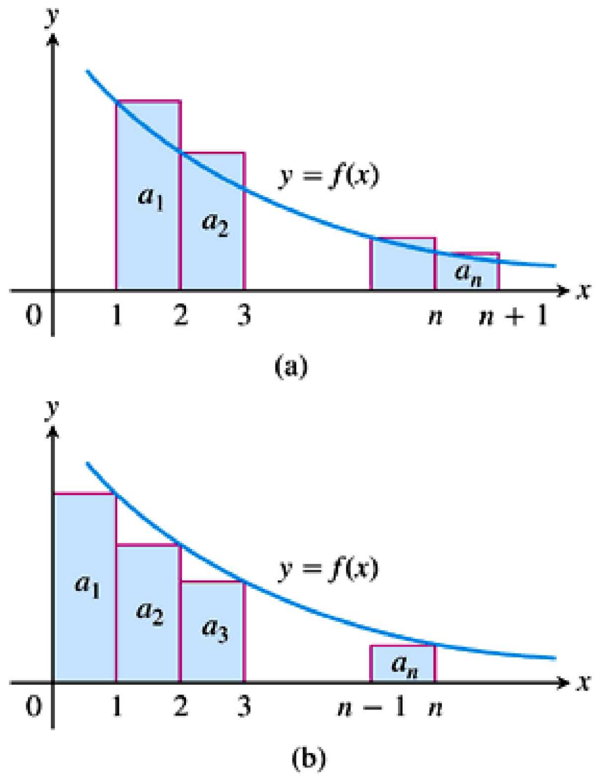
Therefore

$$s_n < 1+\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = 1+\left[-\frac{1}{x}\right]_1^\infty = 2\,.$$

Thus $s_n < 2$ for all n, the partial sums are bounded from above (by 2) and therefore the series converges. Note that the series and the integral need not have the same value in the convergent case.
The approach we have just taken leads us to

THEOREM 9 The Integral Test
Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^\infty f(x)\,dx$ both converge or both diverge.

We will consider the proof for the case $N = 1$ and we start with the asumption that f is a decreasing function with $f(n) = a_n$ for every n.



In part (a) of the above figure, the areas of the rectangles a_1, a_2, $\ldots$, a_n enclose more area than that under the curve $y = f(x)$ between $x = 1$ and $x = n + 1$. Therefore we can write

$$\int_1^{n+1} f(x)\,\mathrm{d}x \le a_1 + a_2 + \cdots + a_n\,.$$

Now consider the rectangles as shown in part (b) above. If we ignore the first rectangle we can write

$$a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,\mathrm{d}x\,.$$

Adding the area a_1 to each side gives

$$a_1 + a_2 + a_3 + \cdots + a_n \le a_1 + \int_1^n f(x)\,\mathrm{d}x$$

Combining the two inequalities gives

$$\int_1^{n+1} f(x)\,\mathrm{d}x \le a_1 + a_2 + \cdots + a_n \le a_1 + \int_1^n f(x)\,\mathrm{d}x\,.$$

These inequalities will hold as $n \to \infty$.
Therefore, if $\int_1^n f(x)\,\mathrm{d}x$ is finite, the right-hand part of the inequality shows that $\sum a_n$ is also finite. Similarly, if $\int_1^{n+1} f(x)\,\mathrm{d}x$ is infinite, then $\sum a_n$ is infinite by the left-hand part of the inequality.

The Integral Test can be used to show that the *p-series* $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $p \le 1$.[1]

[1] See the Thomas' Calculus Section 10.3, p.555 for a proof.

Example:
Show that the series $\sum_{n=1}^{\infty} 1/(n^2+1)$ converges by the integral test.
The function $f(x) = 1/(x^2+1)$ is positive, continuous and decreasing for $x \geq 1$. Also

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x &= \lim_{b\to\infty} [\arctan x]_1^b = \lim_{b\to\infty} [\arctan b - \arctan 1] \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\end{aligned}$$

and so the series converges (but we do not know its sum).

The Ratio Test

THEOREM 12—The Ratio Test Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then **(a)** the series *converges* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

A proof of the above results is given in the textbook.
The two series we looked at in the last section are good examples of cases where $\rho = 1$ and the test is inconclusive:

$$\sum \frac{1}{n} \quad : \quad \frac{a_{n+1}}{a_n} = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \to 1 \quad (n\to\infty)$$

$$\sum \frac{1}{n^2} \quad : \quad \frac{a_{n+1}}{a_n} = \frac{1/(n+1)^2}{1/n^2} = \left(\frac{n}{n+1}\right)^2 \to 1^2 = 1 \quad (n\to\infty)\,.$$

In each case $\rho = 1$ (i.e. the test is inconclusive) and yet we know that $\sum 1/n$ diverges whereas $\sum 1/n^2$ converges.

Example:
Use the Ratio Test to investigate the convergence of the following series:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{2^n+5}{3^n}, \quad \text{(b)} \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}, \quad \text{(c)} \sum_{n=1}^{\infty} \frac{n!}{n^n}\,.$$

(a)

$$a_n = \frac{2^n+5}{3^n}; \qquad a_{n+1} = \frac{2^{n+1}+5}{3^{n+1}};$$

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1}+5)/3^{n+1}}{(2^n+5)/3^n} = \frac{1}{3}\cdot\frac{2^{n+1}+5}{2^n+5} = \frac{1}{3}\left(\frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}}\right)$$

$\to \frac{1}{3}\cdot\frac{2}{1} = \frac{2}{3} < 1$ as $n\to\infty$ and the series converges.

(b)

$$
\begin{aligned}
a_n &= \frac{(2n)!}{(n!)^2}; \qquad a_{n+1} = \frac{(2(n+1))!}{((n+1)!)^2}; \\
\frac{a_{n+1}}{a_n} &= \frac{(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{n!\,n!}{(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \\
= \frac{4n+2}{n+1} &= \frac{4+2/n}{1+1/n} \to 4 > 1 \text{ and the series diverges.}
\end{aligned}
$$

(c)

$$
\begin{aligned}
a_n &= \frac{n!}{n^n}; \qquad a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}; \\
\frac{a_{n+1}}{a_n} &= \frac{(n+1)!\,n^n}{(n+1)^{n+1}n!} = \frac{(n+1)n^n}{(n+1)^n(n+1)} \\
= \frac{n^n}{(n+1)^n} &= \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+1/n}\right)^n \to \frac{1}{e} < 1 \\
&\text{and the series converges.}
\end{aligned}
$$

As we can see, the Ratio Test is often useful when the terms of a series contain factorials involving n or expressions raised to the power involving n.

Power Series

A **power series** is like an "infinite polynomial", i.e., it is an infinite series in powers of some variable, usually x:

DEFINITIONS Power Series, Center, Coefficients

A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \tag{1}$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \tag{2}$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Such series can be *added, subtracted, multiplied, differentiated and integrated* to give new power series.

Example:
Consider the case where the coefficients in (1) in the definition above are all unity.:

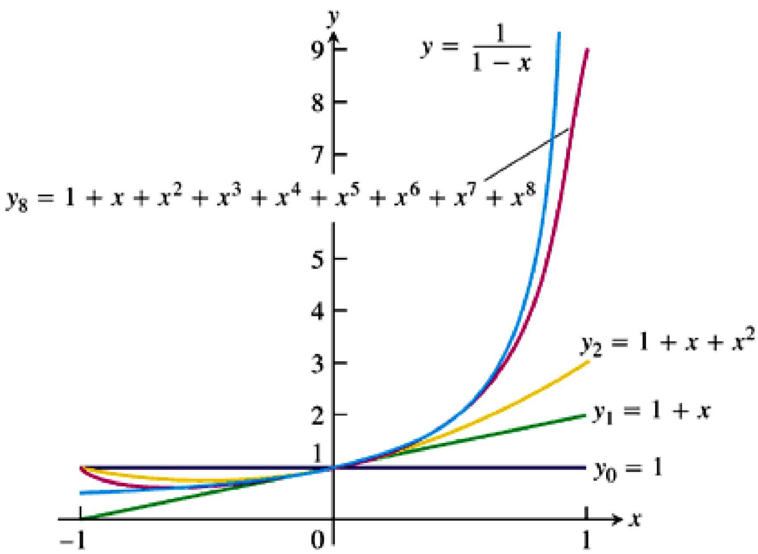
$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots .$$

This is just a geometric series with first term 1 and ratio x ($a = 1, r = x$). We know from the properties of geometric series that it converges to $1/(1-x)$ for $|x| < 1$. Hence

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots\,, \quad -1 < x < 1\,.$$

We can think of the right-hand side of this equation as a sequence of partial sums which are polynomials $P_n(x)$ that *approximate* the function on the left:

$$\begin{array}{llll} f(x) = \dfrac{1}{1-x}; & P_0(x) & = 1 = y_0 & \text{(horizontal line)} \\ & P_1(x) & = 1 + x = y_1 & \text{(straight line, slope 1)} \\ & P_2(x) & = 1 + x + x^2 = y_2 & \text{(quadratic curve [parabola])} \\ & & \vdots & \\ & & \text{etc.} & \end{array}$$



Example:
Consider the power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots\,.$$

This matches the form of (2) in the former definition with $a = 2$, $c_n = (-1/2)^n$. This is a geometric series with the first term 1 and ratio $r = -(x-2)/2$. This series converges for $|(x-2)/2| < 1$ or $0 < x < 4$. The sum is

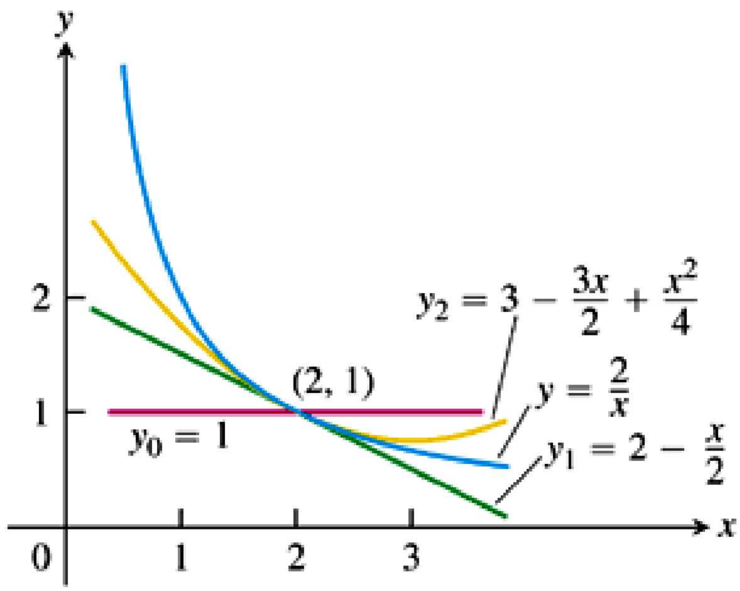
$$\frac{1}{1-r} = \frac{1}{1 + (x-2)/2} = \frac{2}{x}\,.$$

Hence

$$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \cdots + \left(-\frac{1}{2}\right)^2 (x-2)^n + \cdots\,, \quad 0 < x < 4\,.$$

Again we can consider the series as a sequence of partial sums which are polynomials $P_n(x)$ that approximate $2/x$:

$$
\begin{aligned}
f(x) = \frac{2}{x}; \quad P_0(x) &= 1 = y_0 \\
P_1(x) &= 1 - \frac{1}{2}(x-2) = 2 - \frac{x}{2} = y_1 \\
P_2(x) &= 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 = 3 - \frac{3x}{2} + \frac{x^2}{4} = y_2 \\
&\vdots \\
&\text{etc.}
\end{aligned}
$$



A series $\sum a_n$ **converges absolutely** if the corresponding series of absolute values, $\sum |a_n|$, converges. Most importantly, it can be shown that if a series converges *absolutely*, then it *converges*.[2] This enables us to apply the ratio test and the integral test, which only test the convergence of series of positive terms.

A series that converges but does not converge absolutely **converges conditionally**.

THEOREM 18 The Convergence Theorem for Power Series

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges for $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.

[2]See Section 10.6 for a short but tricky proof.

COROLLARY TO THEOREM 18

The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three possibilities:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

Here R is called the **radius of convergence** and the interval of radius R centred at $x = a$ is called the **interval of convergence**.

When studying the convergence of power series such as these, alternating series frequently arise. Here we can make use of an additional test. The **Alternating Series Test** (or Leibniz's Test) states that the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + u_5 - \cdots$$

converges if all three of the following conditions hold:

1. The u_n are all positive,
2. $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N and
3. $u_n \to 0$ as $n \to \infty$.

Example:

The alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

satisfies all of the above three requirements with $N = 1$ and hence converges (but *not* absolutely, as we have shown before).

We can test a power series for convergence using several methods:

1. Use a test such as the *ratio test* to find the interval where the series converges absolutely.
2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint using a test such as the integral test or the alternating sequences test.
3. If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x - a| > R$.

Example:
Use the ratio test to determine the convergence of

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots .$$

We have

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{2n+1}}{2n+1}\frac{2n-1}{x^{2n-1}}\right| = \frac{2n-1}{2n+1}x^2 \to x^2 .$$

Therefore the series converges absolutely for $x^2 < 1$ and diverges for $x^2 > 1$. At $x = 1$ the series is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ which converges by the alternating sequences test. The series also converges at $x = -1$, as can be shown by the alternating sequences test.