MTH4100 Calculus I

Bill Jackson
School of Mathematical Sciences QMUL

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Continuity

Informally a function defined on an interval is continuous if we can sketch its graph in one continuous motion without lifting our pen from the paper. To give a more precise definition we first define what it means for a function to be continuous at a single point in its domain, and to do this we must distinguish between different kinds of points in the domain.

Interior points and end points

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Example: Let $D = [1, 2] \cup (3, 4] \cup \{5\}$. Then D has one left end-point, 1; two right endpoints 2,4; one isolated point 5; and all other points in D are interior points.

Continuity at a point

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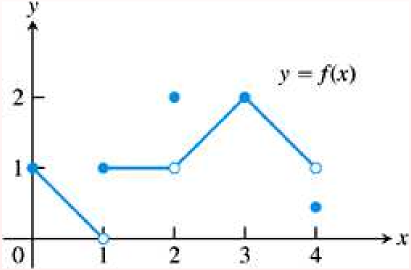
- f is *continuous* at a right end-point b of D if $\lim_{x \to b^-} f(x)$ exists and is equal to $f(b)$.
- f is *continuous* at all isolated point of D.

Example



The function f is continuous at all points in $[0, 4]$ *except at* $x = 1, x = 2$ and $x = 4$.

One-sided continuity at a point

Definition For any (non-isolated) point c in the domain of f we say that:

- f is *right-continuous* at c if $\lim_{x \to c^+} f(x) = f(c)$;
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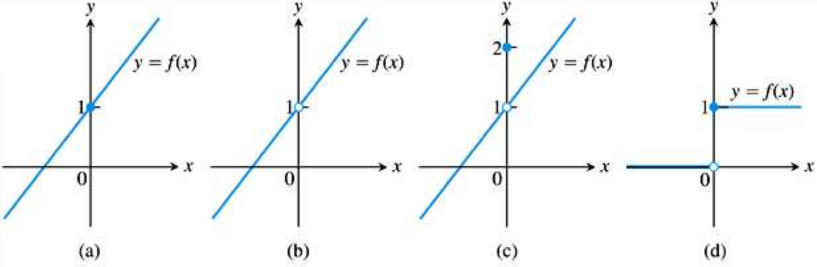
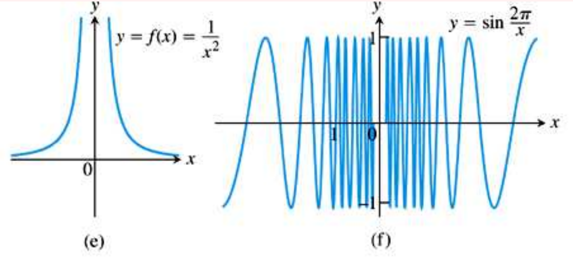
It follows that f is continuous at an interior point c in its domain if and only if it is both right-continuous and left-continuous at c.

Discontinuity at a point

Definition If a function f is not continuous *at a point* $c \in \mathbb{R}$, we say that f is *discontinuous* at c. (Note that f is discontinuous at all points c which do not belong to its domain by definition!)

Examples:


y
y = f(x)
1
0
x
(a)
y
y = f(x)
1
0
x
(b)
y
2
y = f(x)
1
0
x
(c)
y
y = f(x)
1
0
x
(d)


y
y = f(x) = 1/x^2
0
x
(e)
y
y = sin 2π/x
1
−1
0
−1
x
(f)

Properties of continuous functions

The Limit Laws Theorem implies that an algebraic combination of two functions which are both continuous at the same point c, will also be continuous at c.

THEOREM 9 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

Special functions

It is easy to see that the functions $f(x) = x$, and $g(x) = k$ for some constant k, are continuous at c for all $c \in \mathbb{R}$. We can now use the above properties of continuous functions to deduce:

Lemma

All polynomial and rational functions are continuous at c for all $c \in \mathbb{R}$ (provided the denominator of the rational function does not become zero at c).

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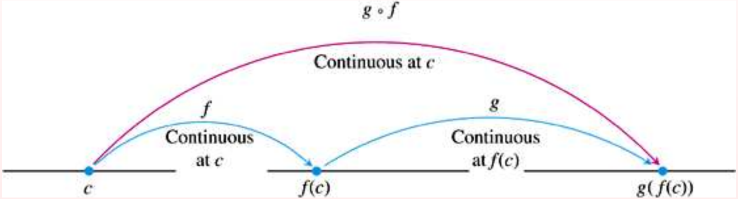
We can also show that trigonometric functions are continuous.

Lemma

The functions $\sin x$ and $\cos x$ are continuous at c for all $c \in \mathbb{R}$. The function $\tan x$ is continuous at c for all $c \in \mathbb{R} \setminus \{\pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots\}$.

Composition of continuous functions

THEOREM 10 Composite of Continuous Functions
If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.



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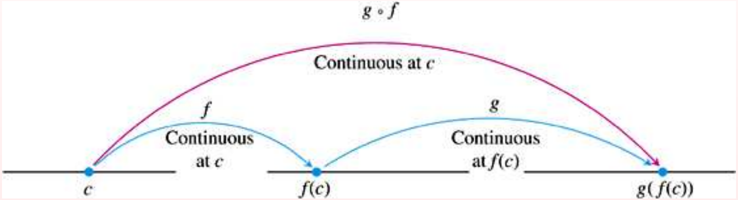

Example: $h(x) = \sin(x^3 + \cos x)$ is continuous at c for all $c \in \mathbb{R}$. This follows since $h = g \circ f$ where $f(x) = x^3 + \cos x$ and $g(x) = \sin x$, and both f and g are continuous at all $c \in \mathbb{R}$.

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Example: We have seen that polynomial, rational and trigonometric functions are all continuous functions.

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A continuous function need not be continuous at all points in $\mathbb{R}$. This will only occur if its domain is equal to $\mathbb{R}$.

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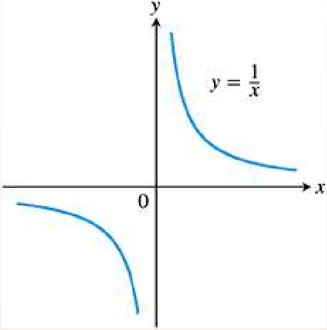
Example: $f(x) = 1/x$.



- f is a continuous function since it is continuous at every point of its domain.
- Nevertheless, f has a **discontinuity** at $x = 0$ since f is not defined at $x = 0$.

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- Deduce that $f(x) = \dfrac{x \sin x}{x^2 + 2}$ is continuous on $(-\infty, \infty)$.

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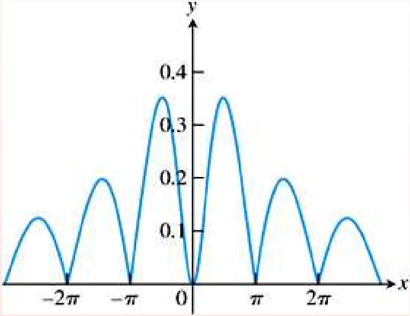
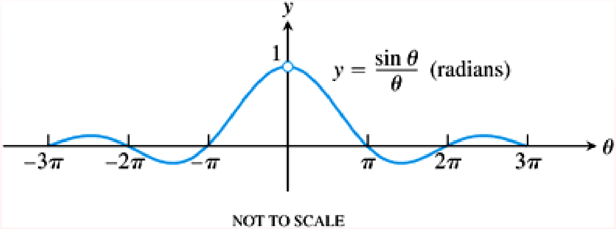


$f(x) = \frac{\sin x}{x}$



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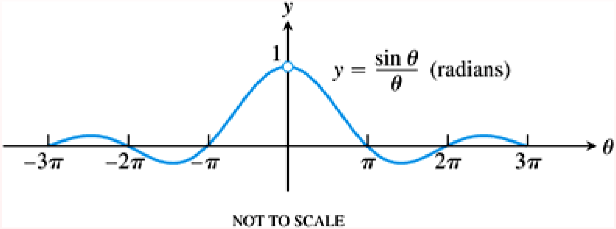


The function f is defined and is continuous at every point $x \in \mathbb{R} \setminus \{0\}$. As $\lim_{x \to 0} \frac{\sin x}{x} = 1$, it makes sense to define a new function F by putting

$$F(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

Then F will be defined and will be continuous at every point $x \in \mathbb{R}$.

Continuous extensions of functions

Definition Suppose $f : D \to \mathbb{R}$ and that $\lim_{x \to c} f(x) = L$ for some $c \in \mathbb{R} \setminus D$. Define a new function $f : D \cup \{c\} \to \mathbb{R}$ by putting

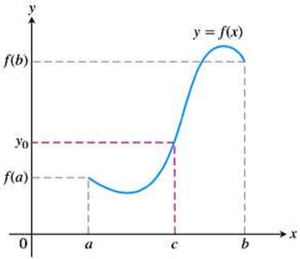
$$F(x) = \begin{cases} f(x) & \text{if } x \neq c \\ L & \text{if } x = c \end{cases}$$

Then F is said to be the *continuous extension of* $f(x)$ *to* c.
Note that F is continuous at c since we have

$$\lim_{x \to c} F(x) = \lim_{x \to c} f(x) = L = F(c).$$

The Intermediate value theorem

THEOREM 11 The Intermediate Value Theorem for Continuous Functions
A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



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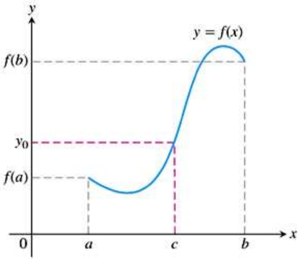
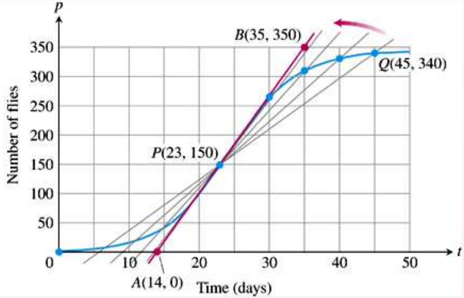


The geometrical interpretation of this theorem is that any horizontal line crossing the y-axis between $f(a)$ and $f(b)$ will cross the graph of $y = f(x)$ at least once over the interval $[a, b]$.

Instantaneous rates of change revisited

Example: Growth of fruit fly population

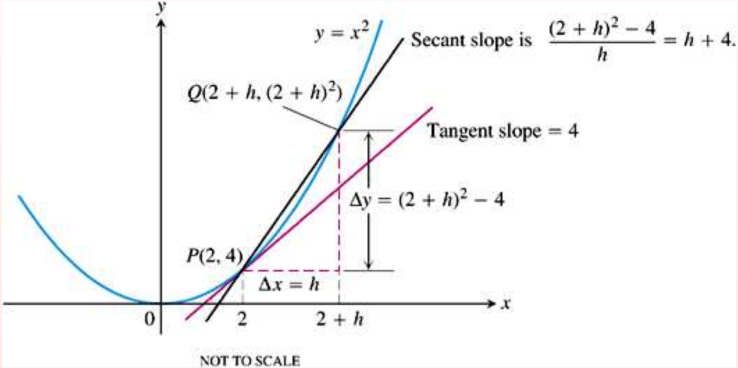
Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\frac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\frac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\frac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\frac{265 - 150}{30 - 23} \approx 16.4$



Basic idea:

- Determine the limit of the slopes of the *secants* QP as Q approaches P.
- Take this limit to be the instantaneous rate of change at P.

Find the equation of the tangent to the parabola $y = x^2$ at the point $P = (2, 4)$.



Definition The *slope* of the curve $y = f(x)$ at the point $P = (x_0, y_0)$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists. The *tangent line* to the curve at P is the line through P with this slope.

Slope and tangent lines

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Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope
$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
3. If the limit exists, find the tangent line as
$$y = y_0 + m(x - x_0).$$

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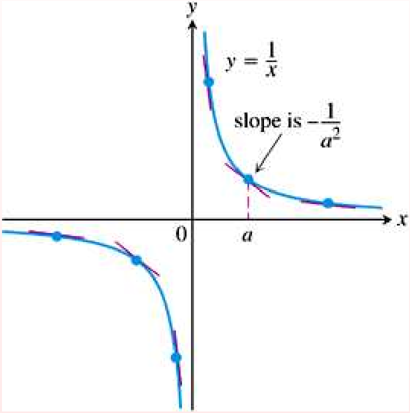

Derivatives

Definition Let $f : D \to \mathbb{R}$. The *derivative* of f is the function f' whose value at a point $c \in D$ is given by

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

provided this limit exists. If $f'(c)$ does exist, then we say that f is *differentiable* at c. If $f'(x)$ exists for all $x \in D$, then we say that the function f is *differentiable*.

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Example Find the derivative of $f(x) = \dfrac{x}{x-1}$.

Alternative formula for the derivative

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Example Differentiate $f(x) = \sqrt{x}$ by using the alternative formula for the derivative.

One-sided derivatives

In analogy to one-sided limits, we can define one-sided derivatives:

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \quad \text{is the } \textit{right-hand derivative} \text{ at } x$$

$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \quad \text{is the } \textit{left-hand derivative} \text{ at } x$$

Then f is differentiable at x if and only if both one-sided derivatives exist and are equal.

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Example: Show that $f(x) = |x|$ is not differentiable at $x = 0$.
[2009 exam question]

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Note: The theorem does imply that if a function is *discontinuous* at $x = c$, then it is *not differentiable* at $x = c$.

Alternative notation for differentiation

We often write $\frac{df}{dx}$ or $\frac{d}{dx}f(x)$ for $f'(x)$.

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The $\frac{d}{dx}$ notation for differentiation was introduced in the late seventeenth century by the German mathematician Gottfried Wilhelm Liebniz and is referred to as *Liebniz notation*.

Rule (Derivative of a Constant Function)

If f is a constant function, $f(x) = c$, then f is differentiable and

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0 \ .$$

Rules for differentiation

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Rule (Power Rule for Positive Integers)

If f is a power function, $f(x) = x^n$ for some $n \in \mathbb{N}$, then f is differentiable and

$$\frac{d}{dx}x^n = nx^{n-1} \ .$$

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Rule (Constant Multiple Rule)

If f is a differentiable function, and c is a constant, then cf is differentiable and

$$\frac{d}{dx}(cf) = c\frac{df}{dx} \ .$$

Rules for differentiation - continued

Rule (Derivative Sum Rule)

If u and v are differentiable functions, then $u + v$ is differentiable and

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}\ .$$

Example: Differentiate $y = 3x^4 + 2$.

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Example: Differentiate $y = (x^2+1)(x^3+3)$.

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Rule (Derivative Quotient Rule)

If u and v are differentiable functions, then u/v is differentiable and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Example: Differentiate $y = \dfrac{t-2}{t^2+1}$.

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Example: Differentiate $y = \dfrac{t-2}{t^2+1}$.

Rule (Power Rule for Negative Integers)

If $f(x) = x^n$ for some negative integer n, then f is differentiable and

$$\frac{d}{dx}x^n = nx^{n-1}\ .$$

Example: $\dfrac{d}{dx}\left(\dfrac{1}{x^{11}}\right) = \dfrac{d}{dx}(x^{-11}) = -11x^{-12}\ .$

Higher-order derivatives

Definition Suppose f is differentiable function. If f' is also differentiable, then we call $f'' = (f')'$ the *second derivative* of f. Similarly, if f'' is differentiable then we we call $f''' = (f'')'$ the *third derivative* of f.

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More generally, if f is differentiable n times for some $n \in \mathbb{N}$ then the *n'th derivative*, $f^{(n)}$, of f is defined recursively by putting $f^{(0)} = f$, and

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Example: Find the first four derivatives of $f(x) = x^3$ and $g(x) = x^{-2}$.