Queen Mary
University of London

# MTH4100 Calculus I

## Lecture notes for Week 5

## Thomas' Calculus, Sections 2.4 to 2.6

Prof Bill Jackson

School of Mathematical Sciences
Queen Mary University of London

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# Continuity

Informally a function defined on an interval is continuous if we can sketch its graph in one continuous motion without lifting our pen from the paper. To give a more precise definition we first define what it means for a function to be continuous at a single point in its domain, and to do this we must distinguish between different kinds of points in the domain.

**Definition** Let $D \subset \mathbb{R}$ and $x \in D$. Then:

- $x$ is an *interior point* of $D$ if we have $x \in I$ for some open interval $I = (a, b) \subseteq D$;
- $x$ is a *left end-point* (respectively *right end-point* of $D$) if $x$ is not an interior point of $D$ and we have $x \in I$ for some half-closed interval $I = [x, b) \subseteq D$ (respectively $I = (a, x] \subseteq D$);
- $x$ is an *isolated point* of $D$ if $x$ is neither an interior point nor an end-point.

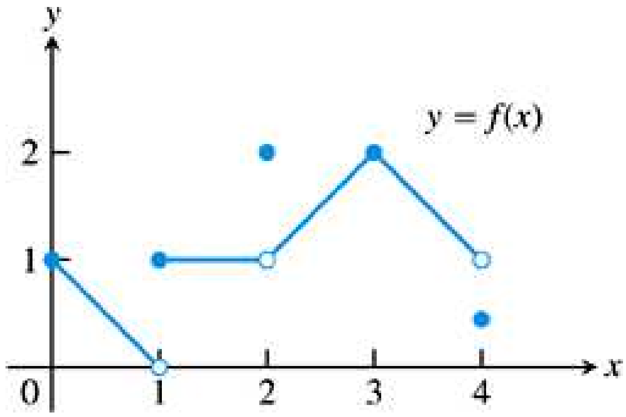
**Example:** Let $D = [1, 2] \cup (3, 4] \cup \{5\}$. Then $D$ has one left end-point, 1; two right endpoints 2,4; one isolated point 5; and all other points in $D$ are interior points.

We can now define continuity at a point:

**Definition** Let $f$ be a function with domain $D \subset \mathbb{R}$. Then:

- $f$ is *continuous* at an interior point $c$ of $D$ if $\lim_{x \to c} f(x)$ exists and is equal to $f(c)$.
- $f$ is *continuous* at a left end-point $a$ of $D$ if $\lim_{x \to a^+} f(x)$ exists and is equal to $f(a)$.
- $f$ is *continuous* at a right end-point $b$ of $D$ if $\lim_{x \to b^-} f(x)$ exists and is equal to $f(b)$.
- $f$ is *continuous* at every isolated point of $D$.[1]

**Example:** $f : [0, 4] \to \mathbb{R}$




The function $f$ is continuous at all points in $[0, 4]$ *except at* $x = 1, x = 2$ and $x = 4$ since:

- $\lim_{x \to 1} f(x)$ does not exist;
- $\lim_{x \to 2} f(x) = 1 \neq f(2)$;
- $\lim_{x \to 4^-} f(x) = 1 \neq f(4)$.

[1] In this module our domains will never have isolated points so this part of the definition will never be used.

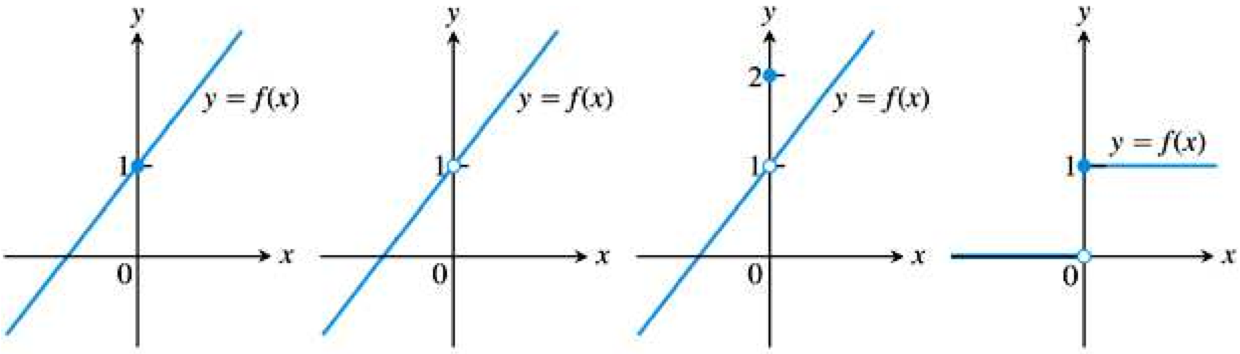
We can also define 'one-sided continuity'. For any (non-isolated) point $c$ in the domain of $f$ we say that:

- $f$ is *right-continuous* at $c$ if $\lim_{x \to c^+} f(x) = f(c)$;
- $f$ is *left-continuous* at $c$ if $\lim_{x \to c^-} f(x) = f(c)$;

It follows that $f$ is continuous at an interior point $c$ in its domain if and only if it is both right-continuous and left-continuous at $c$.

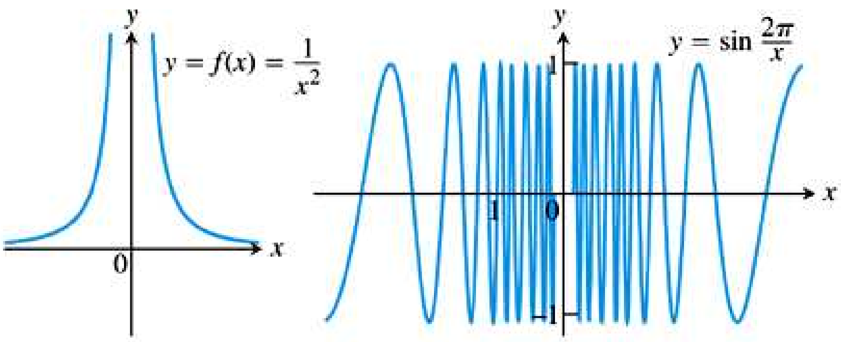
If a function $f$ is not continuous *at a point* $c \in \mathbb{R}$, we say that $f$ is *discontinuous* at $c$. Note that $f$ is discontinuous at all points $c$ which do not belong to its domain by definition.

**Examples:** Continuity and discontinuity at $x = 0$.




(a) (b) (c) (d)

continuous — not continuous — jump discontinuity




(e) (f)

infinite discontinuity — oscillating discontinuity

**Note** We can easily repair the discontinuity at $x = 0$ in cases (b) and (c) be (re)defining $f(0)$ as in (a). There is no easy way to repair the discontinuity at $x = 0$ in (d), (e), and (f).

The Limit Laws Theorem implies that an algebraic combination of two functions which are both continuous at the same point $c$, will also be continuous at $c$.

**THEOREM 9 Properties of Continuous Functions**

If the functions $f$ and $g$ are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number $k$
5. *Quotients*: $f/g$ provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing $c$, where $r$ and $s$ are integers

**Remark:** It is easy to see that the functions $f(x) = x$, and $g(x) = k$ for some constant $k$, are continuous at $c$ for all $c \in \mathbb{R}$. We can now use the above properties of continuous functions to deduce that all polynomial and rational functions are continuous at $c$ for all $c \in \mathbb{R}$ (provided the denominator of the rational function does not become zero at $c$). We can also show that trigonometric functions are continuous.

**Lemma 1** *The functions* $\sin x$ *and* $\cos x$ *are continuous at* $c$ *for all* $c \in \mathbb{R}$. *The function* $\tan x$ *is continuous at* $c$ *for all* $c \in \mathbb{R} \setminus \{\pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots\}$.

**Proof** We have

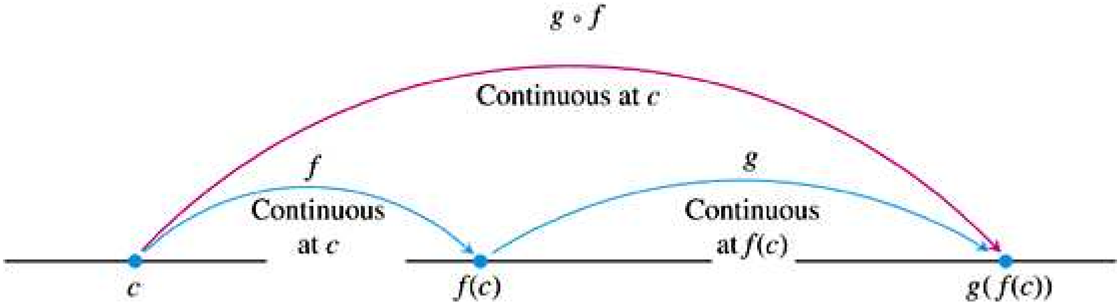
$$\begin{aligned}
\lim_{x\to c} \sin x &= \lim_{h\to 0} \sin(c+h) && \text{[substituting } h = x - c\text{]} \\
&= \lim_{h\to 0}(\sin c \cos h + \cos c \sin h) \\
&= \sin c \lim_{h\to 0}(\cos h) + \cos c \lim_{h\to 0}(\sin h) \\
&= \sin c && \text{[since } \lim_{h\to 0}(\cos h) = 1 \text{ and } \lim_{h\to 0}(\sin h) = 0\text{]}
\end{aligned}$$

A similar proof works for $\cos x$ (Check this!). We can now deduce that $\tan x$ is continuous at $x = c$ whenever $\cos c \neq 0$ by using $\tan x = \sin x / \cos x$.

We next state a result which says that compositions of continuous functions are continuous.

**THEOREM 10 Composite of Continuous Functions**

If $f$ is continuous at $c$ and $g$ is continuous at $f(c)$, then the composite $g \circ f$ is continuous at $c$.

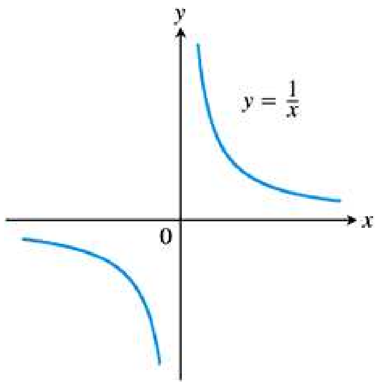



**Example:** $h(x) = \sin(x^3 + \cos x)$ is continuous at $c$ for all $c \in \mathbb{R}$. This follows since $h = g \circ f$ where $f(x) = x^3 + \cos x$ and $g(x) = \sin x$, and both $f$ and $g$ are continuous at $c$ for all $c \in \mathbb{R}$.

**Definition** A function $f$ is *continuous on an interval* $I$ if $f$ is continuous at every point of $I$. Similarly $f$ is said to be a *continuous function* if $f$ is continuous at every point of its domain.

**Example:** We have seen that polynomial, rational and trigonometric functions are all continuous functions.

Note that a continuous function need not be continuous at all points in $\mathbb{R}$. This will only occur if its domain is equal to $\mathbb{R}$.
**Example:** $f(x) = 1/x$.

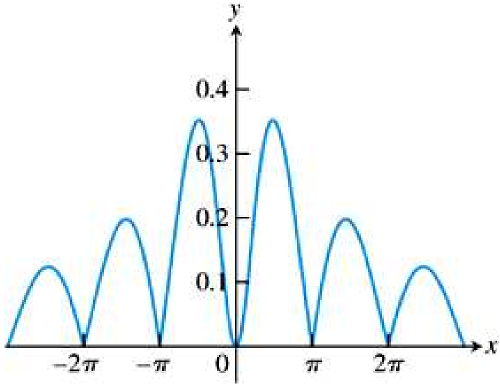



- $f$ is a continuous function since it is continuous at every point of its domain.
- Nevertheless, $f$ has a **discontinuity** at $x = 0$ since $f$ is not defined at $x = 0$.

**Example:** Show that $h(x) = \left|\dfrac{x \sin x}{x^2 + 2}\right|$ is continuous on $(-\infty, \infty)$.

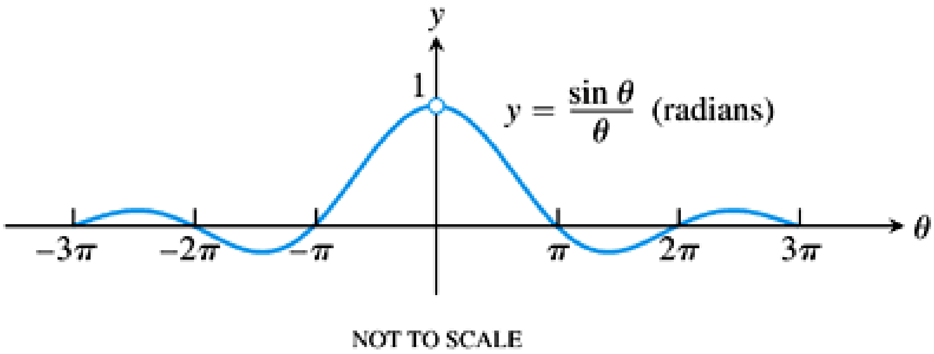
- Note that $y = \sin x$ is continuous on $(-\infty, \infty)$.
- Deduce that $f(x) = \dfrac{x \sin x}{x^2 + 2}$ is continuous on $(-\infty, \infty)$.
- Show that $g(x) = |x|$ is continuous on $(-\infty, \infty)$.
- Deduce that $h = g \circ f$ is continuous on $(-\infty, \infty)$.

$y = \left|\dfrac{x \sin x}{x^2+2}\right|$




### Continuous extensions of functions

**Example:** $f(x) = \frac{\sin x}{x}$




The function $f$ is defined and is continuous at every point $x \in \mathbb{R} \setminus \{0\}$. As $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, it makes sense to define a new function $F$ by putting

$$F(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

Then $F$ will be defined and will be continuous at every point $x \in \mathbb{R}$.
**Definition** Suppose $f : D \to \mathbb{R}$ and that $\lim\limits_{x\to c} f(x) = L$ for some $c \in \mathbb{R} \setminus D$. Define a new function $f : D \cup \{c\} \to \mathbb{R}$ by putting

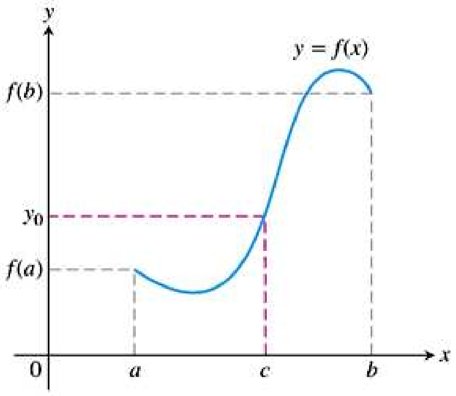
$$F(x) = \begin{cases} f(x) & \text{if } x \neq c \\ L & \text{if } x = c \end{cases}$$

Then $F$ is said to be the *continuous extension of* $f(x)$ *to* $c$. (Note that $F$ is continuous at $c$ since we have $\lim_{x\to c} F(x) = \lim_{x\to c} f(x) = L = F(c)$.

### The Intermediate value theorem

This result tells us that whenever a continuous function takes on two values, it must take on all the values in between.

**THEOREM 11 The Intermediate Value Theorem for Continuous Functions**
A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if $y_0$ is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some $c$ in $[a, b]$.



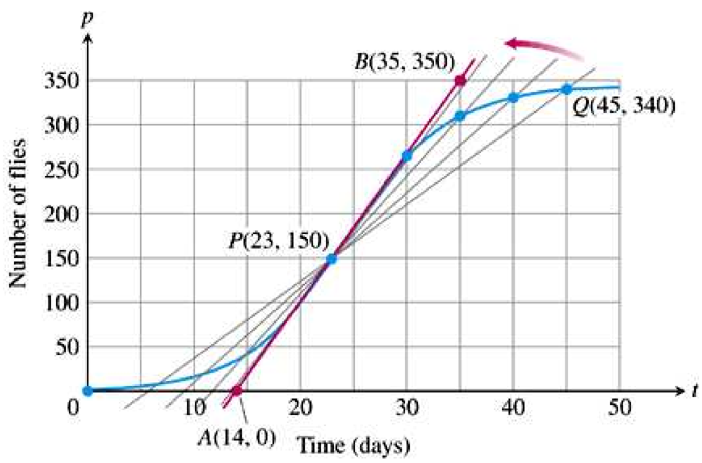

The geometrical interpretation of this theorem is that any horizontal line crossing the $y$-axis between $f(a)$ and $f(b)$ will cross the graph of $y = f(x)$ at least once over the interval $[a, b]$. Note that continuity is essential: if $f$ is discontinuous at some point in the interval, then the function may "jump" and miss some values.

## Differentiation

Recall our discussion of average and instantaneous rates of change.

**Example:** Growth of fruit fly population

| $Q$ | Slope of $PQ = \Delta p / \Delta t$ (flies/day) |
|---|---|
| (45, 340) | $\frac{340 - 150}{45 - 23} \approx 8.6$ |
| (40, 330) | $\frac{330 - 150}{40 - 23} \approx 10.6$ |
| (35, 310) | $\frac{310 - 150}{35 - 23} \approx 13.3$ |
| (30, 265) | $\frac{265 - 150}{30 - 23} \approx 16.4$ |




Basic idea:

- Determine the limit of the slopes of the secants[2] $QP$ as $Q$ approaches $P$.

[2]In this context, a *secant* is a line joining two points of a curve.

- Take this limit to be the instantaneous rate of change at $P$.

**Example:** Find the equation of the tangent to the parabola $y = x^2$ at the point $P = (2, 4)$.

- Choose a point $Q = (2 + h, (2 + h)^2)$ on the parabola a **horizontal distance** $h \neq 0$ away from $P$.
- The secant $PQ$ has slope

$$\frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 2^2}{(2+h) - 2} = \frac{4 + 4h + h^2 - 4}{h} = 4 + h\ .$$

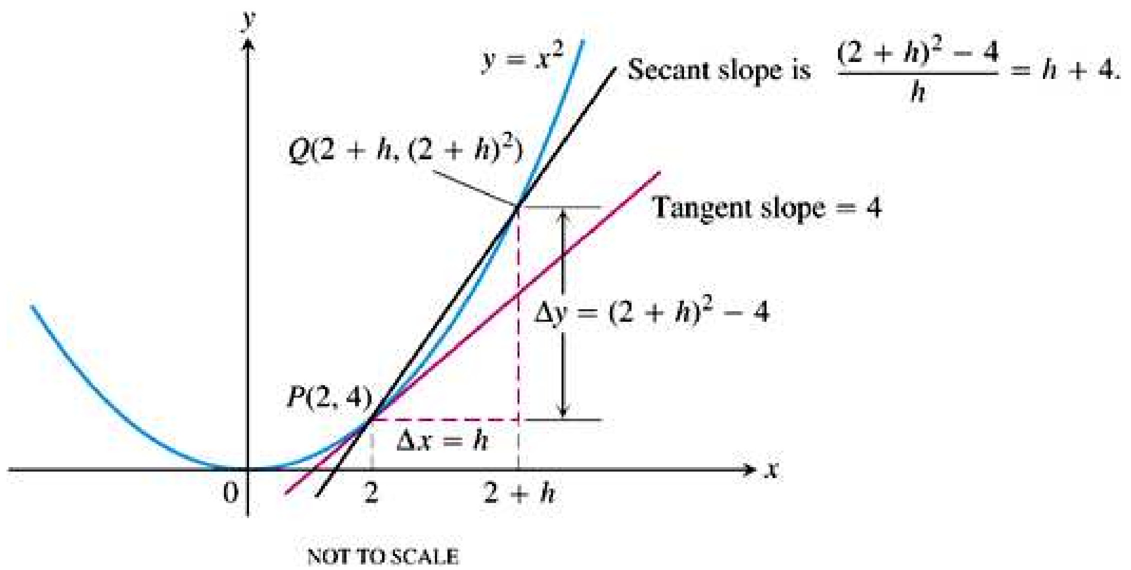
- As $Q$ approaches $P$, $h$ approaches 0. Hence

$$m = \lim_{h \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} (4 + h) = 4$$

  is the parabola's slope at $P$.

- The equation of the tangent through $P$ is $y = y_1 + m(x - x_1)$ where $P = (x_1, y_1) = (2, 4)$ and $m = 4$. This gives $y = 4 + 4(x - 2) = 4x - 4$.

**Summary:**




This approach generalises to arbitrary curves and arbitrary points:

**Definition** The *slope* of the curve $y = f(x)$ at the point $P = (x_0, y_0)$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

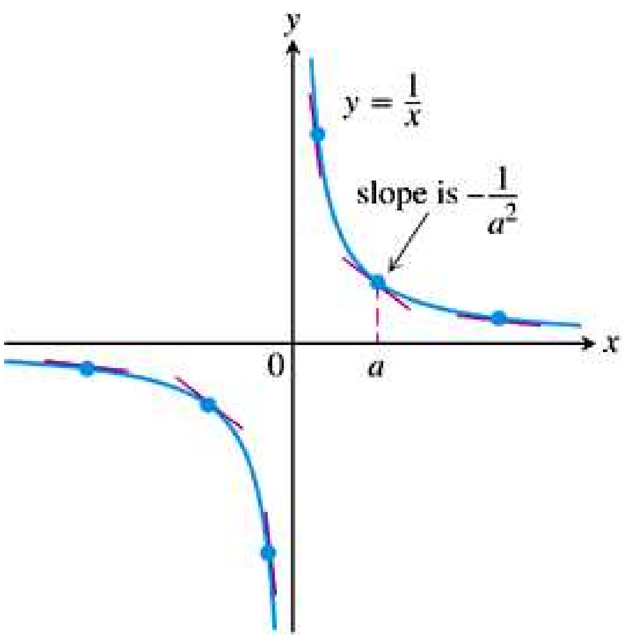
provided this limit exists. The *tangent line* to the curve at $P$ is the line through $P$ with this slope.

**Finding the Tangent to the Curve $y = f(x)$ at $(x_0, y_0)$**

1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope
$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
3. If the limit exists, find the tangent line as
$$y = y_0 + m(x - x_0).$$

**Example:** Find slope and tangent to $y = 1/x$ at $x = a$ when $a \neq 0$

1. $f(a) = \frac{1}{a}$, $f(a+h) = \frac{1}{a+h}$
2. slope:
$$\begin{aligned} m &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} \\ &= \lim_{h \to 0} \frac{a - (a+h)}{h \cdot a(a+h)} \\ &= \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2} \end{aligned}$$
3. tangent line at $(a, 1/a)$: $y = \frac{1}{a} + \left(-\frac{1}{a^2}\right)(x - a) = \frac{2}{a} - \frac{x}{a^2}$.



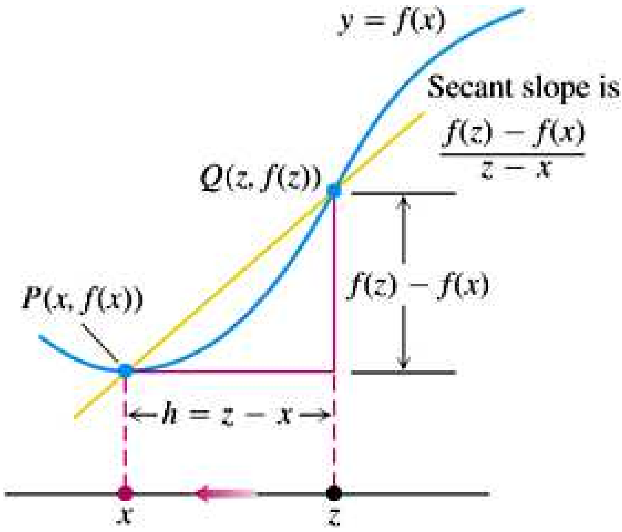
**Definition** Let $f : D \to \mathbb{R}$. The *derivative* of $f$ is the function $f'$ whose value at a point $c \in D$ is given by

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

provided this limit exists. If $f'(c)$ does exist, then we say that $f$ is *differentiable* at $c$. If $f'(x)$ exists for all $x \in D$, then we say that the function $f$ is *differentiable.*

We can obtain an alternative formula for $f'(x)$ by putting $z = x + h$. Then $z \to x$ as $h \to 0$ and we have

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$




**Alternative notations:** We often write $\frac{df}{dx}$ or $\frac{d}{dx}f(x)$ for $f'(x)$. Furthermore, if $y = f(x)$ then we write $y'$ or $\frac{dy}{dx}$ instead of $f'(x)$.[3]

Calculating a derivative is called *differentiation* ("derivation" is something else!).

**Example:** Differentiate from first principles $f(x) = \frac{x}{x-1}$.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \\
&= \lim_{h \to 0} \frac{(x+h)(x-1) - x(x+h-1)}{h(x+h-1)(x-1)} \\
&= \lim_{h \to 0} \frac{-h}{h(x+h-1)(x-1)} \\
&= -\frac{1}{(x-1)^2}
\end{aligned}$$

[3]The $\frac{d}{dx}$ notation for differentiation was introduced in the late seventeenth century by the German mathematician Gottfried Wilhelm Liebniz and is referred to as *Liebniz notation.*

**Example:** Differentiate $f(x) = \sqrt{x}$ by using the alternative formula for derivatives.

$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\ &= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\ &= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \\ &= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

**One-sided derivatives**: In analogy to one-sided limits, we can define one-sided derivatives:

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \quad \text{is the } \textit{right-hand derivative} \text{ at } x$$

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \quad \text{is the } \textit{left-hand derivative} \text{ at } x$$

Then:

$f$ is differentiable at $x$ if and only if both one-sided derivatives exist and are equal.

**Example:** Show that $f(x) = |x|$ is not differentiable at $x = 0$. [2009 exam question]

- right-hand derivative at $x = 0$:

$$\lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} 1 = 1$$

- left-hand derivative at $x = 0$:

$$\lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} (-1) = -1 \ .$$

Since the right-hand and left-hand derivatives differ the limit does not exist.

**Theorem 1** *If $f$ has a derivative at $x = c$, then $f$ is continuous at $x = c$.*

**Proof:** *Trick:* For $h \neq 0$, we have

$$f(c+h) = f(c) + \frac{f(c+h) - f(c)}{h} h \ .$$

By assumption, $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$. We also have $\lim_{h \to 0} f(c) = f(c)$ and $\lim_{h \to 0} h = 0$. Hence, by the Limit Laws,

$$\lim_{h \to 0} f(c+h) = f(c) + f'(c) \cdot 0 = f(c) \ .$$

Thus $f$ is continuous at $x = c$. •

**Caution:** The converse of this theorem is *false*! Consider for example $f(x) = |x|$. This function is continuous at $x = 0$ but is not differentiable at $x = 0$.
**Note:** The theorem does imply that if a function is *discontinuous* at $x = c$, then it is *not differentiable* at $x = c$.

### Rules for Differentiation

The following rules are useful for working out derivatives. We will prove one them. See Thomas, Section 3.2, for proofs of the others.

**Rule 1 (Derivative of a Constant Function)** *If $f$ is a constant function, $f(x) = c$, then $f$ is differentiable and*

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0 \ .$$

**Rule 2 (Power Rule for Positive Integers)** *If $f$ is a power function, $f(x) = x^n$ for some $n \in \mathbb{N}$, then $f$ is differentiable and*

$$\frac{d}{dx}x^n = nx^{n-1} \ .$$

**Rule 3 (Constant Multiple Rule)** *If $f$ is a differentiable function, and $c$ is a constant, then $cf$ is differentiable and*

$$\frac{d}{dx}(cf) = c\frac{df}{dx} \ .$$

**Rule 4 (Derivative Sum Rule)** *If $u$ and $v$ are differentiable functions, then $u + v$ is differentiable and*

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \ .$$

**Example:** Differentiate $y = 3x^4 + 2$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(3x^4+2) \\ &= \frac{d}{dx}(3x^4) + \frac{d}{dx}(2) \qquad \text{(by rule 4)} \\ &= 3\frac{d}{dx}(x^4) + 0 \qquad \text{by rules 1,3)} \\ &= 3 \cdot 4x^3 \qquad \text{(by rule 2)} \\ &= 12x^3 \end{aligned}$$

**Rule 5 (Derivative Product Rule)** *If $u$ and $v$ are differentiable functions, then $uv$ is differentiable and*

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \ .$$

**Proof** We have

$$\begin{aligned}\frac{u(x+h)v(x+h)-u(x)v(x)}{h} &= \frac{u(x+h)v(x+h)-u(x+h)v(x)+u(x+h)v(x)-u(x)v(x)}{h} \\ &= \frac{u(x+h)[v(x+h)-v(x)]}{h}+\frac{v(x)[u(x+h)-u(x)]}{h}\end{aligned}$$

Since $u$ and $v$ are differentiable, $\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}=\frac{du}{dx}$ and $\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}=\frac{dv}{dx}$. Since $u$ is differentiable, it is continuous and hence $\lim_{h\to 0}u(x+h)=u(x)$. We also have $\lim_{h\to 0}u(x)=u(x)$. The Limit Laws now give

$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{h\to 0}\frac{u(x+h)v(x+h)-u(x)v(x)}{h} \\ &= \lim_{h\to 0}u(x+h)\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}+\lim_{h\to 0}v(x)\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} \\ &= u(x)\frac{dv}{dx}+v(x)\frac{du}{dx}\end{aligned}$$

•

**Example:** Differentiate $y=(x^2+1)(x^3+3)$.
Let $u=x^2+1$ and $v=x^3+3$. Then $u'=2x$ and $v'=3x^2$. Hence

$$y'=uv'+vu'=(x^2+1)3x^2+2x(x^3+3)=5x^4+3x^2+6x.$$

**Rule 6 (Derivative Quotient Rule)** *If $u$ and $v$ are differentiable functions, then $u/v$ is differentiable and*

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}\,.$$

**Example:** Differentiate $y=\dfrac{t-2}{t^2+1}$.
Let $u=t-2$ and $v=t^2+1$. Then $u'=1$ and $v'=2t$. Hence

$$y'=\frac{1(t^2+1)-(t-2)2t}{(t^2+1)^2}=\frac{-t^2+4t+1}{(t^2+1)^2}$$

**Warning:** $(uv)'\neq u'v'$ and $(u/v)'\neq u'/v'$.

**Rule 7 (Power Rule for Negative Integers)** *If $f(x)=x^n$ for some negative integer $n$, then $f$ is differentiable and*

$$\frac{d}{dx}x^n=nx^{n-1}\,.$$

**Example:** $\dfrac{d}{dx}\left(\dfrac{1}{x^{11}}\right)=\dfrac{d}{dx}(x^{-11})=-11x^{-12}\,.$

## Higher-order derivatives

**Definition** Suppose $f$ is differentiable function. If $f'$ is also differentiable, then we call $f''=(f')'$ the *second derivative* of $f$. Similarly, if $f''$ is differentiable then we we call

$f''' = (f'')'$ the *third derivative* of $f$. More generally, if $f$ is differentiable $n$ times for some $n \in \mathbb{N}$ then the *n'th derivative*, $f^{(n)}$, of $f$ is defined recursively by putting $f^{(0)} = f$, and

$$f^{(n)} = \frac{df^{(n-1)}}{dx}$$

for $n \geq 1$.

**Example:** Find the first four derivatives of $f(x) = x^3$ and $g(x) = x^{-2}$.

$$\begin{aligned} f'(x) &= 3x^2 & g'(x) &= -2x^{-3} \\ f''(x) &= 6x & g''(x) &= 6x^{-4} \\ f'''(x) &= 6 & g'''(x) &= -24x^{-5} \\ f^{(4)}(x) &= 0 & g^{(4)}(x) &= 120x^{-6}. \end{aligned}$$