Queen Mary
University of London

MTH4101 Calculus II

Lecture notes for Week 5
Series I and II

Thomas' Calculus, Sections 10.1 to 10.3

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Spring 2013

Example:

Does the sequence whose nth term is $a_n = ((n+1)/(n-1))^n$ converge? If so, find $\lim_{n\to\infty} a_n$. If we just take the straightforward limit we get the indeterminate form 1^∞. Typically with questions of this type we take the logarithm. This gives:

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n = n\,\ln\left(\frac{n+1}{n-1}\right).$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} \ln a_n &= \lim_{n\to\infty} n\ln\left(\frac{n+1}{n-1}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \\
&= \lim_{n\to\infty} \frac{\ln(n+1) - \ln(n-1)}{1/n} \\
&= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} \quad \text{(using l'Hôpital's Rule)} \\
&= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2\,.
\end{aligned}$$

Let $b_n = \ln a_n$ Then $\lim_{n\to\infty} b_n = 2$ and since $f(x) = e^x$ is continuous we have by the continuous function theorem for sequences

$$a_n = e^{\ln a_n} = e^{b_n} \to e^2 \quad \text{as} \quad n \to \infty\,.$$

Therefore the sequence $\{a_n\}$ converges to e^2.

The following Theorem summarizes some common results for the limits of sequences:

THEOREM 5

The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

The first result can be proved using l'Hôpital's rule. The second and third results can be proved using logarithms and applying the previous theorems. Proofs of the remaining results are given in Appendix 5 of Thomas' Calculus.

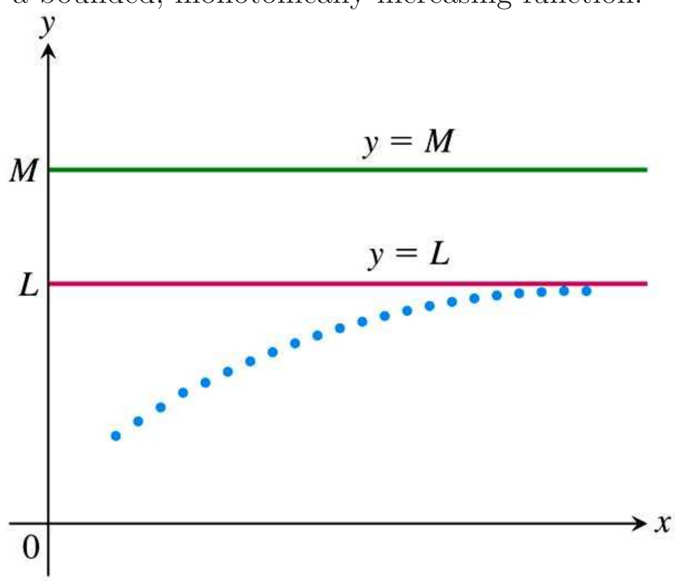
Example:
Show that $\lim_{n\to\infty} \sqrt[n]{n^2} = 1$.

$$\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} n^{2/n} = \lim_{n\to\infty} \left(n^{1/n}\right)^2 = (1)^2 = 1\,.$$

For *bounded, monotonic* sequences there is the following theorem:

THEOREM 6—The Monotonic Sequence Theorem If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

For example, look at a bounded, monotonically increasing function:



Example:

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1\,.$$

Infinite series

An **infinite series** is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots\,.$$

Example:

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots\,.$$

DEFINITIONS **Infinite Series, *n*th Term, Partial Sum, Converges, Sum**

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

A *geometric series* has the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$$

where a and r are fixed real numbers and $a \neq 0$. The quantity r is called the *ratio* of the geometric series and can be positive or negative.
In the special case where $r = 1$ the nth partial sum is

$$s_n = a + a \cdot 1 + a \cdot 1^2 + \cdots + a \cdot 1^{n-1} = na$$

and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$ depending on the sign of a. If $r = -1$ the series diverges because either $s_n = a$ or $s_n = 0$ depending on the value of n.
Now consider the case of a geometric series with $|r| \neq 1$. We have

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ s_n - rs_n &= a - ar^n \quad \text{or} \quad s_n(1-r) = a(1-r^n) \\ \Rightarrow s_n &= \frac{a(1-r^n)}{1-r} \quad (r \neq 1)\,. \end{aligned}$$

Therefore, if $|r| < 1$ then $r^n \to 0$ as $n \to \infty$ and hence $s_n \to a/(1-r)$. If $|r| > 1$ then $|r^n| \to \infty$ and the series diverges. So we have

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{for} \quad |r| < 1$$

and the geometric series converges.
For example,

$$\frac{1}{9}+\frac{1}{27}+\frac{1}{81}\cdots=\sum_{n=1}^{\infty}\frac{1}{9}\left(\frac{1}{3}\right)^{n-1}=\frac{(1/9)}{1-(1/3)}=\frac{1}{6}\qquad(a=1/9,\ r=1/3)$$

and

$$5-\frac{5}{4}+\frac{5}{16}-\frac{5}{64}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n 5}{4^n}=\frac{5}{1+(1/4)}=4\qquad(a=5,\ r=-1/4)\,.$$

Example:
Find the sum of the series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\,.$$

Note that we can use partial fractions to write

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}\,.$$

Hence the sum of the first k terms is

$$\sum_{n=1}^{k}\frac{1}{n(n+1)}=\sum_{n=1}^{k}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

and so the kth partial sum is

$$\begin{aligned} s_k &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{k}-\frac{1}{k+1}\right)\\ &= \frac{1}{1}+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\cdots+\left(-\frac{1}{k}+\frac{1}{k}\right)-\frac{1}{k+1}\end{aligned}$$

Hence $s_k \to 1$ as $k\to\infty$ and so the series converges giving

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1\,.$$

Suppose the series $\sum_{n=1}^{\infty} a_n$ converges to a sum S and the nth partial sum of the series is $s_n = a_1+a_2+\cdots+a_n$. When n is large, both s_n and s_{n-1} are close to S and therefore their difference a_n is close to zero. Using the Difference Rule for sequences we have

$$a_n = s_n - s_{n-1}\quad\longrightarrow\quad S-S=0\quad\text{as}\quad n\to\infty\,.$$

Hence:

THEOREM 7

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

This, in turn, leads to

The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Example:

$$\sum_{n=1}^{\infty} n^2 \quad \text{diverges because} \quad n^2 \to \infty$$
$$\sum_{n=1}^{\infty} \frac{n+1}{n} \quad \text{diverges because} \quad \frac{n+1}{n} \to 1$$
$$\sum_{n=1}^{\infty} (-1)^{n+1} \quad \text{diverges because} \quad \lim_{n\to\infty} (-1)^{n+1} \quad \text{does not exist}$$
$$\sum_{n=1}^{\infty} \frac{-n}{2n+5} \quad \text{diverges because} \quad \lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0\,.$$

Note that the *converse of the above theorem is false*: If $a_n \to 0$ this does **not** imply that the series $\sum_{n=1}^{\infty} a_n$ converges.

Example:
Consider the unusual case of a series where $a_n \to 0$ but the series itself diverges:

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n} + \cdots$$

where there are two terms of $1/2$, four terms of $1/4$, ..., 2^n terms of $1/2^n$, etc. In this case each grouping of terms adds up to 1 so the partial sums must increase without bound and so the series diverges, even though the terms of the series form a sequence that converges to 0.

If we have two convergent series, we can add them term by term, subtract them term by term, or multiply them by constants to make new convergent series:

THEOREM 8
If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum k a_n = k \sum a_n = kA$ (Any number k).

Example:
Find $\sum_{n=1}^{\infty}(3^{n-1}-1)/6^{n-1}$.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{3^{n-1}-1}{6^{n-1}} &= \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}-\frac{1}{6^{n-1}}\right)=\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}-\sum_{n=1}^{\infty}\frac{1}{6^{n-1}} \\ &= \frac{1}{1-(1/2)}-\frac{1}{1-(1/6)} \quad \text{(two geometric series)} \\ &= 2-\frac{6}{5}=\frac{4}{5}\,.\end{aligned}$$

We can add a finite number of terms or delete a finite number of terms without altering the convergence or divergence of a series but if the series is convergent this will usually alter the sum. Consider the series

$$\sum_{n=1}^{\infty}a_n = a_1+a_2+\cdots+a_{k-1}+\sum_{n=k}^{\infty}a_n\,.$$

If $\sum_{n=1}^{\infty}a_n$ converges, then $\sum_{n=k}^{\infty}a_n$ converges for any $k>1$. Conversely, if $\sum_{n=k}^{\infty}a_n$ converges for any $k>1$, then $\sum_{n=1}^{\infty}a_n$ converges.
Note that re-indexing a series (e.g. changing the starting value of the index) does not alter its convergence, provided the order of the terms is preserved.
For example, raise the starting value of the index h units:

$$n=k-h: \quad \sum_{n=1}^{\infty}a_n=\sum_{k=1+h}^{\infty}a_{k-h}=a_1+a_2+a_3+\cdots\,.$$

Lower the starting value of the index h units:

$$n=k+h: \quad \sum_{n=1}^{\infty}a_n=\sum_{k=1-h}^{\infty}a_{k+h}=a_1+a_2+a_3+\cdots\,.$$

The Integral Test

For a given series $\sum a_n$ we want to know: (1) Does it converge? (2) If it converges, what is its sum?
A corollary of the Monotonic Sequence Theorem is that the series $\sum_{n=1}^{\infty}a_n$ of *non-negative* terms converges *if and only if* (why?) its partial sums are bounded from above.
As a warm-up, consider the **harmonic series**:

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots\,.$$

This series is actually divergent even though the nth term $1/n\to 0$ as $n\to\infty$, cf. the *n-th term test* seen before. However, the series has no upper bound for its partial sums. We can see this by writing the series as

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right)+\cdots\,.$$

Now $\frac{1}{3}+\frac{1}{4}>\frac{2}{4}=\frac{1}{2}$, $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{4}{8}=\frac{1}{2}$, $\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}>\frac{8}{16}=\frac{1}{2}$ and so on. Therefore the sum of the 2^n terms ending with $1/2^{n+1}$ is $> 2^n/2^{n+1} = 1/2$. Therefore the sequence of partial sums is not bounded from above, and so the harmonic series diverges. Now consider the series,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{n^2}+\cdots$$

Does it converge or diverge? To answer this question we will consider a new approach involving the use of integration. What we need to do is compare series $\sum_{n=1}^{\infty} 1/n^2$ with the integral $\int_1^{\infty} 1/x^2\,\mathrm{d}x$.