MTH4100 Calculus I

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One-sided limits

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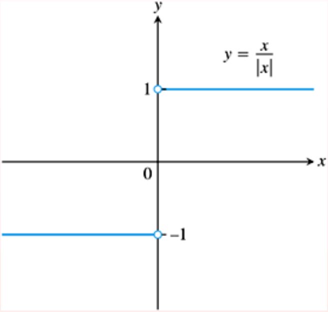
Similarly, *M is a right-hand limit of f at c* if $f(x)$ becomes arbitrarily close to M as x approaches c from above. We write

$$\lim_{x \to c^+} f(x) = M.$$

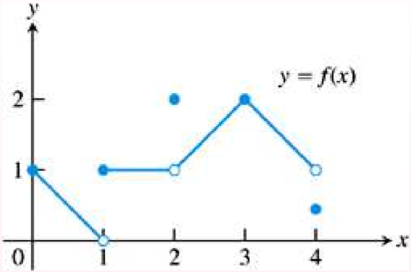
Example

$$f(x) = \frac{x}{|x|}.$$



- $\lim_{x \to 0^+} f(x) = 1$
- $\lim_{x \to 0^-} f(x) = -1$
- $\lim_{x \to 0} f(x)$ **does not exist**

Example



c	$\lim_{x\to c^-} f(x)$	$\lim_{x\to c^+} f(x)$	$\lim_{x\to c} f(x)$
0	cannot exist	1	cannot exist
1	0	1	does not exist
2	1	1	1
3	2	2	2
4	1	cannot exist	cannot exist

Results

Theorem

A function f has a limit at c if and only if it has both a left-hand and right-hand limit at c and these two limits are equal, i.e.

$$\lim_{x\to c} f(x) = L \text{ if and only if } \lim_{x\to c^-} f(x) = L \text{ and } \lim_{x\to c^+} f(x) = L$$

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The Limit Law Theorem and results about limits of polynomials and rational functions also hold for one-sided limits.

The Sandwich Theorem

Theorem (The Sandwich Theorem)

Suppose that f, g, h are functions defined on an open interval I containing c (except possibly at c itself). Suppose further that $g(x) \leq f(x) \leq h(x)$ for all $x \in I \setminus \{c\}$ and that $\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x)$. Then

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The sandwich theorem can be used to calculate the limit of a complicated function when its values are 'sandwiched between' those of two simpler functions. In particular we can use it to determine limits of trigonometric functions.

Limits of trigonometric functions

Lemma

$\lim_{\theta \to 0} \sin\theta = 0$ *and* $\lim_{\theta \to 0} \cos\theta = 1$.

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Example Determine $\lim_{\theta\to 0}\dfrac{\cos\theta - 1}{\theta}.$

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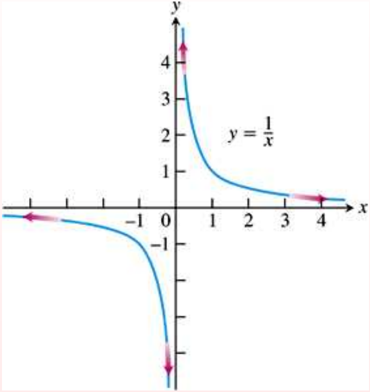

We would like to describe the behavior of $f(x)$ as $|x|$ gets very large.

Limits at infinity

Informal definition We say that $f(x)$ *has the limit* L *as* x *approaches infinity* and write

$$\lim_{x \to \infty} f(x) = L$$

if, as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L. Similarly, we say that $f(x)$ *has the limit* L *as* x *approaches minus infinity* and write

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Examples:

$$\lim_{x\to\infty} k = k = \lim_{x\to-\infty} k$$

and

$$\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to-\infty} \frac{1}{x}.$$

Limit Laws

Theorem (Limit laws as x approaches infinity)

Suppose that L, M are real numbers, and f and g are functions such that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} g(x) = M$. Then

1. Sum Rule: $\lim\limits_{x\to\infty}(f(x) + g(x)) = L + M$
   The limit of the sum of two functions is the sum of their limits.
2. Difference Rule: $\lim\limits_{x\to\infty}(f(x) - g(x)) = L - M$
3. Constant Multiple Rule: $\lim\limits_{x\to\infty}(kf(x)) = kL$ *for any constant* $k \in \mathbb{R}$.
4. Product Rule: $\lim\limits_{x\to\infty}(f(x)g(x)) = LM$
5. Quotient Rule: $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ *when* $M \neq 0$
6. Power Rule: $\lim\limits_{x\to\infty}(f(x))^{r/s} = L^{r/s}$ *for any integers* r, s *such that* $L^{r/s}$ *is a real number.*

Horizontal Asymptotes

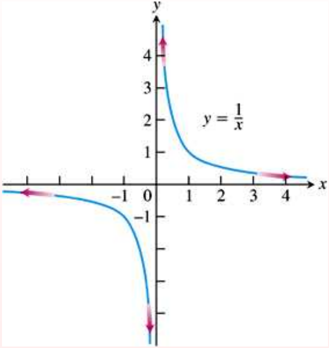
Limits for $f(x)$ as x approaches $\pm\infty$ give rise to 'horizontal asymptotes'.

DEFINITION Horizontal Asymptote

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = b.$$

Example



We have

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0$$

This tells us that the graph of $y = 1/x$ approaches the line $y = 0$ as $|x|$ becomes very large. Thus the line $y = 0$ is a horizontal asymptote of the graph.

Horizontal Asymptotes

Example Calculate the horizontal asymptote(s) for the graph of $y = \frac{5x^2+8x-3}{3x^2+2}$.

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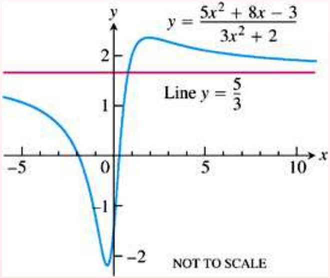

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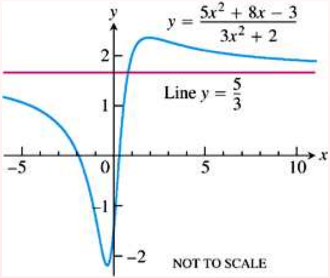

A similar approach will give us the horizontal asymptotes of any rational function in which the degree of the numerator is less than or equal to the degree of the denominator: we divide both the numerator and denominator by the largest power of x appearing in the denominator.

Oblique asymptotes

How does a rational function $f(x) = p(x)/q(x)$ behave as $|x|$ gets large when the degree of $p(x)$ is one greater than the degree of $q(x)$?

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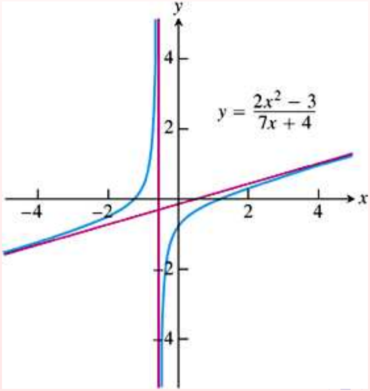


Oblique asymptotes

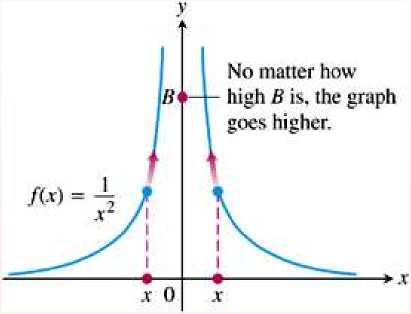
In general, if a rational function $f(x) = p(x)/q(x)$ has the degree of $p(x)$ one greater than the degree of $q(x)$, then polynomial division gives

$$f(x) = ax + b + r(x) \text{ with } \lim_{x \to \infty} r(x) = 0 = \lim_{x \to -\infty} r(x)$$

In this case the line $y = ax + b$ is said to be an *oblique (or slanted) asymptote* of $f(x)$.

Infinite limits - Example

What is the behaviour of $f(x) = \dfrac{1}{x^2}$ near $x = 0$?



Informal definition We say that $f(x)$ *approaches infinity as* x *approaches* x_0 and write

$$\lim_{x \to x_0} f(x) = \infty$$

if the values of $f(x)$ grow without bound as x approaches x_0, eventually reaching and surpassing every positive real number.

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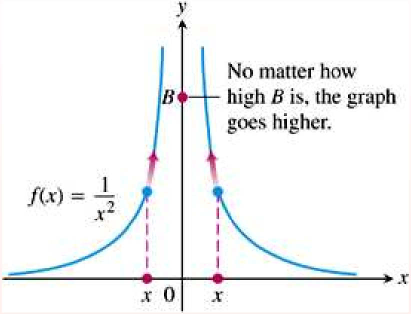
if the values of $f(x)$ decrease without bound as x approaches x_0, eventually reaching and surpassing every negative real number.

Infinite limits - Example continued

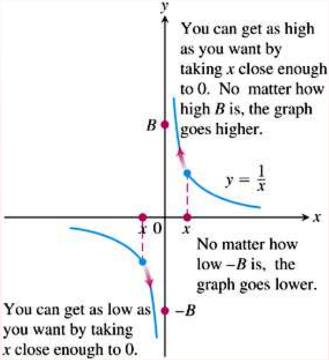


$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

as the values of $1/x^2$ are positive and become arbitrarily large as x approaches 0 from the right or the left.

One sided infinite limits - Example

$f(x) = 1/x$



We say that $f(x)$ *approaches infinity as* x *approaches* 0 *from the right* and write $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

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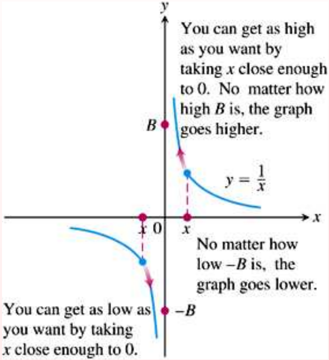


We say that $f(x)$ *approaches infinity as* x *approaches* 0 *from the right* and write $\lim_{x\to 0^+} \frac{1}{x} = \infty$.
Similarly, we say that $f(x)$ *approaches minus infinity as* x *approaches* 0 *from the left* and write $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

Vertical asymptotes

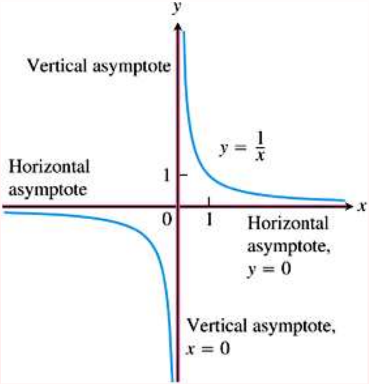
Infinite limits give rise to 'vertical asymptotes' in the graph of a function:

DEFINITION **Vertical Asymptote**

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

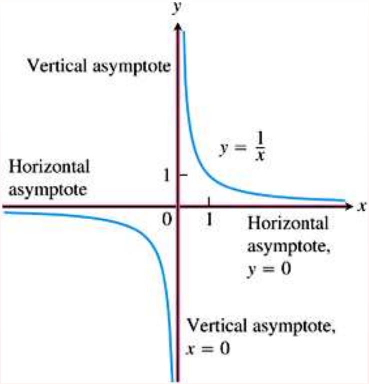
$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

Vertical asymptotes - Example



Since $\lim_{x \to 0^+} \frac{1}{x} = \infty$ and $\lim_{x \to 0^-} \frac{1}{x} = -\infty$, the graph of $y = 1/x$ approaches the line $x = 0$ as x approaches 0, and this line is a vertical asymptote of the graph.

Vertical asymptotes - Example



Since $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$, the graph of $y = 1/x$ approaches the line $x = 0$ as x approaches 0, and this line is a vertical asymptote of the graph.

The graph of $y = 1/x$ has two asymptotes: the line $y = 0$ is a horizontal asymptote and the line $x = 0$ is a vertical asymptote.

Vertical asymptotes - Example

Find the asymptotes of

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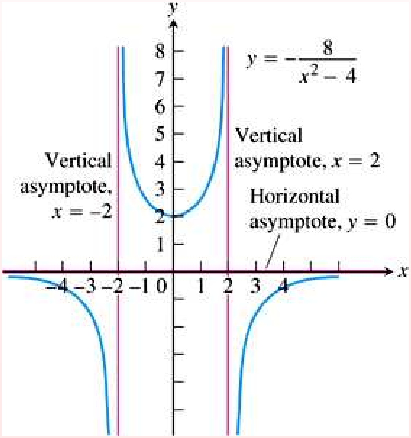

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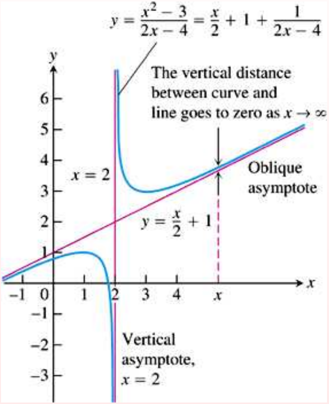

We say that the term $\frac{x}{2} + 1$ dominates $f(x)$ when $|x|$ is large and that the term $\frac{1}{2x-4}$ dominates $f(x)$ when x is close to 2.

Informally a function defined on an interval is continuous if we can sketch its graph in one continuous motion without lifting our pen from the paper. To give a more precise definition we first define what it means for a function to be continuous at a single point in its domain, and to do this we must distinguish between different kinds of points in the domain.

Interior points and end points

Definition Let $D \subset \mathbb{R}$ and $x \in D$. Then:

- x is an *interior point* of D if we have $x \in I$ for some open interval $I = (a, b) \subseteq D$;

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- x is a *left end-point*, respectively *right end-point*, of D if x is not an interior point of D and we have $x \in I$ for some half-closed interval $I = [x, b) \subseteq D$, respectively $I = (a, x] \subseteq D$;

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- x is an *isolated point* of D if x is neither an interior point nor an end-point.

Example: Let $D = [1, 2] \cup (3, 4] \cup \{5\}$. Then D has one left end-point, 1; two right endpoints 2,4; one isolated point 5; and all other points in D are interior points.

Continuity at a point

Definition Let f be a function with domain $D \subset \mathbb{R}$. Then:

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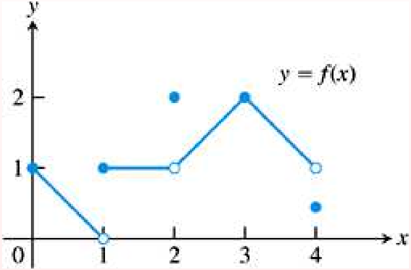
- f is *continuous* at all isolated point of D.

Example



The function f is continuous at all points in $[0, 4]$ *except at* $x = 1, x = 2$ and $x = 4$.