MTH4100 Calculus I

Lecture notes for Week 4

Thomas' Calculus, Sections 2.4 to 2.6

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One-sided limits

For a function f to have the limit L as x approaches c, $f(x)$ must become arbitrarily close to L as x approaches c from *both* sides. But we can also consider the behavior of $f(x)$ as x approaches c from only one of the two sides.

Informal Definition We say that L *is a left-hand limit of* f *at* c if $f(x)$ becomes arbitrarily close to L as x approaches c from below and write

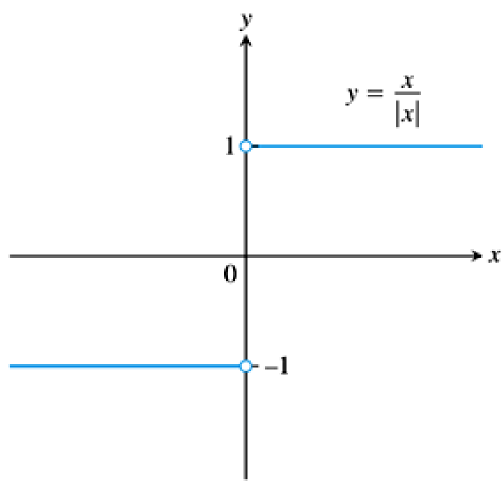
$$\boxed{\lim_{x\to c^-} f(x) = L}.$$

Similarly, we say that M *is a right-hand limit of* f *at* c if $f(x)$ becomes arbitrarily close to M as x approaches c from above and write

$$\boxed{\lim_{x\to c^+} f(x) = M}.$$

The symbol $x \to c^+$ means that we only consider values of x *greater than* c. The symbol $x \to c^-$ means that we only consider values of x *less than* c.

Example:



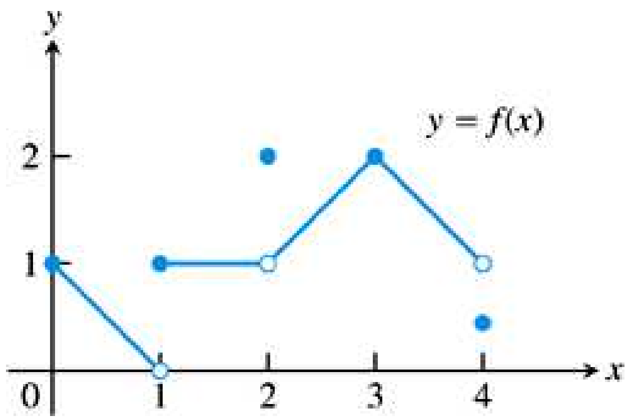
- $\lim_{x\to 0^+} f(x) = 1$
- $\lim_{x\to 0^-} f(x) = -1$
- $\lim_{x\to 0} f(x)$ **does not exist**

Theorem 1 *A function f has a limit at c if and only if it has both a left-hand and right-hand limit at c and these two limits are equal, i.e.*

$$\lim_{x\to c} f(x) = L \text{ if and only if } \lim_{x\to c^-} f(x) = L \text{ and } \lim_{x\to c^+} f(x) = L$$

The Limit Law Theorem and results about limits of polynomials and rational functions all hold for one-sided limits.

Example:



c	$\lim_{x\to c^-} f(x)$	$\lim_{x\to c^+} f(x)$	$\lim_{x\to c} f(x)$
0	cannot exist	1	cannot exist
1	0	1	does not exist
2	1	1	1
3	2	2	2
4	1	cannot exist	cannot exist

The 'cannot exist' entries follow because f is not defined in a suitable open interval for us to be able to apply the definition for the corresponding limit. For example the definition of limit at $x = 0$ requires f to be defined on an open interval containing 0 (except possibly at 0 itself). This is not the case since $f(x)$ is not defined when $x < 0$.

The Sandwich Theorem and limits of trigonometric functions

Theorem 2 (The Sandwich Theorem) *Suppose that f, g, h are functions defined on an open interval I containing c (except possibly at c itself). Suppose further that $g(x) \leq f(x) \leq h(x)$ for all $x \in I \setminus \{c\}$ and that $\lim_{x\to c} g(x) = L = \lim_{x\to c} h(x)$. Then $\lim_{x\to c} f(x) = L$.*

A similar result holds for one-sided limits.

The sandwich theorem can be used to calculate the limit of a complicated function when its values are 'sandwiched between' those of two simpler functions. In particular we can use it to determine limits of trigonometric functions.

Corollary 1

$$\lim_{\theta\to 0} \sin\theta = 0 \ \textit{and} \ \lim_{\theta\to 0} \cos\theta = 1.$$

Proof These limits follow from the sandwich theorem using the facts that

$$-|\theta| \leq \sin\theta \leq |\theta|$$

and

$$-|\theta| \leq 1 - \cos\theta \leq |\theta|$$

for all $\theta \in \mathbb{R}$, see Thomas' Calculus, page 53, Example 11. •

Corollary 2

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

Proof We can use the one sided version of the Sandwich Theorem, Corollary 1 and the fact that

$$\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1$$

for all $\theta \in (0, \pi/2)$ to deduce that

$$\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1.$$

The fact that $\frac{\sin \theta}{\theta}$ is an even function of θ now implies that we also have $\lim_{\theta \to 0^-} \frac{\sin \theta}{\theta} = 1$. Theorem 1 now gives

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

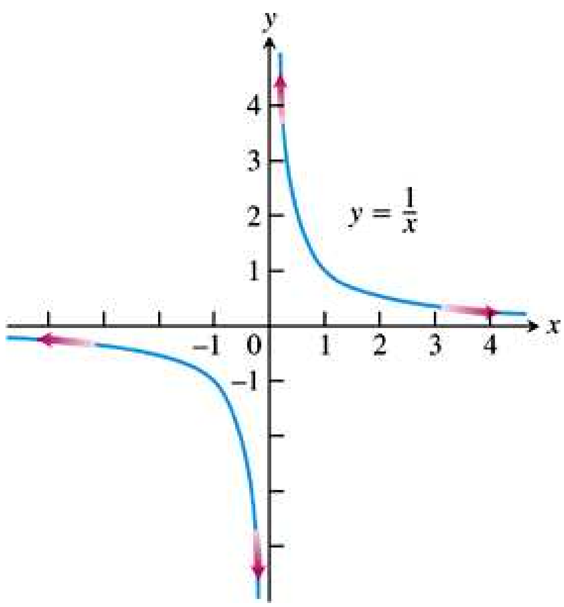
See Thomas' Calculus, page 70, Theorem 7 for more details. •

Example: Compute $\lim_{h \to 0} \frac{\cos h - 1}{h}$. We have

$$\begin{aligned}
\lim_{h \to 0} \frac{\cos h - 1}{h} &= \lim_{h \to 0} \frac{-2\sin^2(h/2)}{h} \qquad [\text{since } \sin^2(h/2) = (1 - \cos h)/2] \\
&= \lim_{h \to 0} -\frac{\sin(h/2)}{h/2} \sin(h/2) \\
&= \lim_{\theta \to 0} -\frac{\sin \theta}{\theta} \sin \theta \qquad [\text{by substituting } \theta = h/2] \\
&= -1 \cdot 0 = 0 \qquad [\text{by Corollaries 1 and 2 and the Limit Laws}]
\end{aligned}$$

Limits at infinity and asymptotes

Example:



We would like to describe the behavior of $f(x)$ as $|x|$ gets very large.

Informal definition We say that $f(x)$ *has the limit* L *as* x *approaches infinity* and write

$$\lim_{x\to\infty} f(x) = L$$

if, as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L. Similarly, we say that $f(x)$ *has the limit* L *as* x *approaches minus infinity* and write

$$\lim_{x\to-\infty} f(x) = L$$

if, as x moves increasingly far from the origin in the negative direction, $f(x)$ gets arbitrarily close to L.

Examples:

$$\lim_{x\to\infty} k = k = \lim_{x\to-\infty} k$$

and

$$\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to-\infty} \frac{1}{x}.$$

The Limit Laws Theorem remains valid if we replace $\lim_{x\to c}$ by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$. For example:

Theorem 3 (Limit laws as x approaches infinity) *Suppose that* L, M *are real numbers, and* f *and* g *are functions such that* $\lim_{x\to\infty} f(x) = L$ *and* $\lim_{x\to\infty} g(x) = M$. *Then*

1. Sum Rule: $\lim_{x\to\infty}(f(x)+g(x)) = L + M$
   The limit of the sum of two functions is the sum of their limits.
2. Difference Rule: $\lim_{x\to\infty}(f(x)-g(x)) = L - M$
3. Constant Multiple Rule: $\lim_{x\to\infty}(kf(x)) = kL$ *for any constant* $k \in \mathbb{R}$.
4. Product Rule: $\lim_{x\to\infty}(f(x)g(x)) = LM$
5. Quotient Rule: $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{L}{M}$ *when* $M \neq 0$
6. Power Rule: $\lim_{x\to\infty}(f(x))^{r/s} = L^{r/s}$ *for any integers* r, s *such that* $L^{r/s}$ *is a real number.*

Example:

$$\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} = 5 + 0 = 5.$$

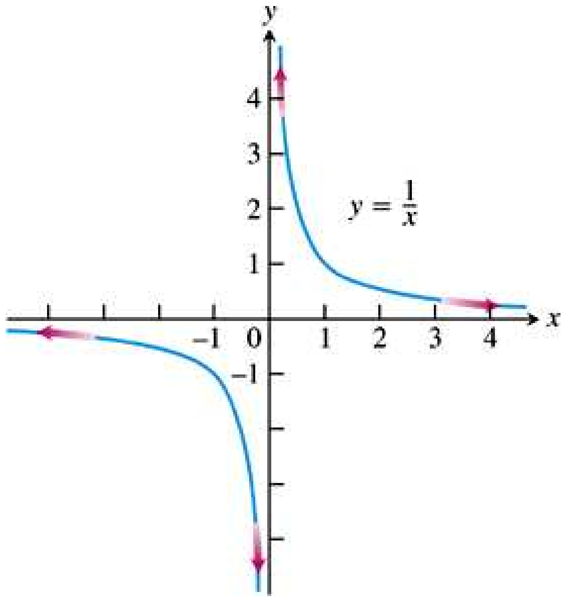
Limits for $f(x)$ as x approaches $\pm\infty$ give rise to 'horizontal asymptotes'.

DEFINITION Horizontal Asymptote

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

Example:



We have

$$\lim_{x\to\infty}\frac{1}{x}=0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{1}{x}=0$$

This tells us that the graph of $y = 1/x$ approaches the line $y = 0$ as $|x|$ becomes very large. Thus the line $y = 0$ is a horizontal asymptote of the graph.

Example: Calculate the horizontal asymptote(s) for the graph of $y = \frac{5x^2+8x-3}{3x^2+2}$.

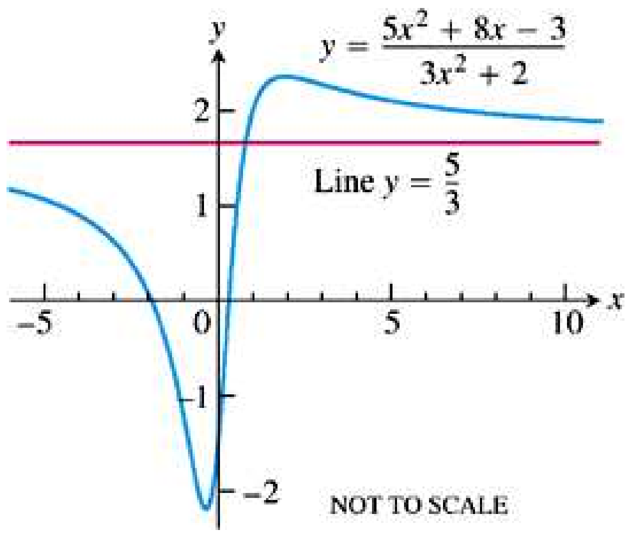
We have:

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to\infty}\frac{x^2(5+8/x-3/x^2)}{x^2(3+2/x^2)}=\lim_{x\to\infty}\frac{5+8/x-3/x^2}{3+2/x^2}=\frac{5}{3}$$

Similarly

$$\lim_{x\to-\infty}\frac{5x^2+8x-3}{3x^2+2}=\frac{5}{3}$$

Thus the graph of f has the line $y = 5/3$ as a horizontal asymptote on both the left and the right.



A similar approach will give us the horizontal asymptotes of any rational function in which the degree of the numerator is less than or equal to the degree of the denominator: we divide both the numerator and denominator by the largest power of x appearing in the denominator.

What happens if the degree of the polynomial in the numerator is one greater than that in the denominator?

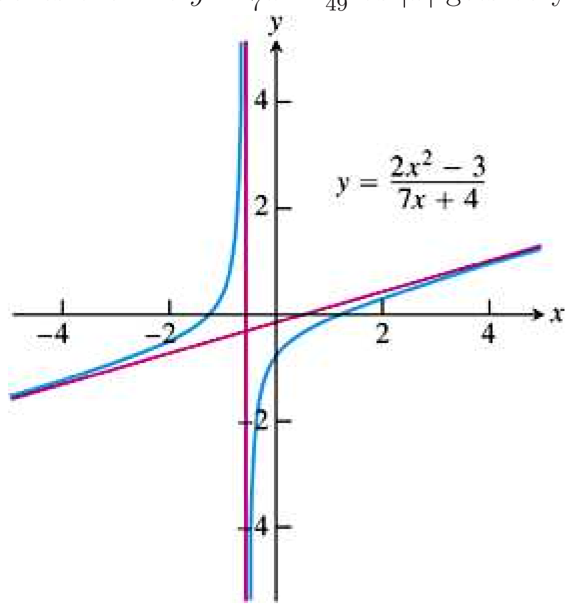
Example: Consider $f(x) = \frac{2x^2-3}{7x+4}$. We have

$$f(x) = \frac{2x^2-3}{7x+4} = \frac{2}{7}x - \frac{8}{49} - \frac{115}{49(7x+4)}$$

for all $x \neq -4/7$. Since

$$\lim_{x\to\infty} \frac{-115}{49(7x+4)} = 0 = \lim_{x\to-\infty} \frac{-115}{49(7x+4)}$$

the graph of $f(x)$ approaches the line $y = \frac{2}{7}x - \frac{8}{49}$ as $|x|$ gets very large.



In general, if a rational function $f(x) = p(x)/q(x)$ has the degree of $p(x)$ one greater than the degree of $q(x)$, then polynomial division gives

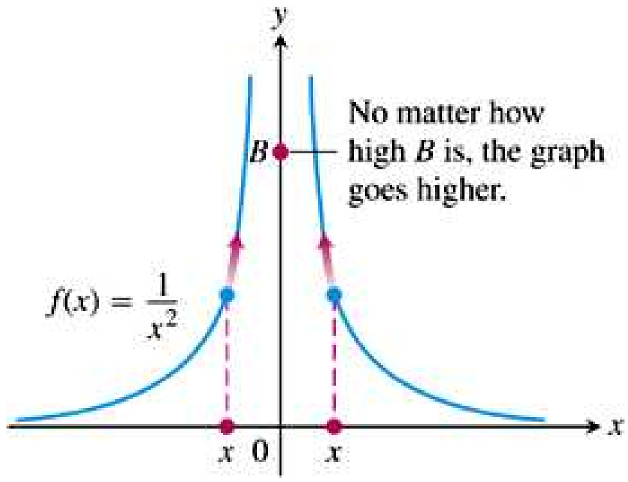
$$f(x) = ax + b + r(x) \qquad \text{with} \lim_{x\to\pm\infty} r(x) = 0$$

In this case the line $y = ax + b$ is said to be an *oblique (or slanted) asymptote* of $f(x)$.

In the above example, the line $y = \frac{2}{7}x - \frac{8}{49}$ is an oblique asymptote of $f(x)$.

Infinite limits and vertical asymptotes

Example: What is the behaviour of $f(x) = \dfrac{1}{x^2}$ near $x = 0$?



Informal definition We say that $f(x)$ *approaches infinity as* x *approaches* x_0 and write

$$\lim_{x \to x_0} f(x) = \infty$$

if the values of $f(x)$ grow without bound as x approaches x_0, eventually reaching and surpassing every positive real number. Similarly, we say that $f(x)$ *approaches negative infinity as* x *approaches* x_0 and write

$$\lim_{x \to x_0} f(x) = -\infty$$

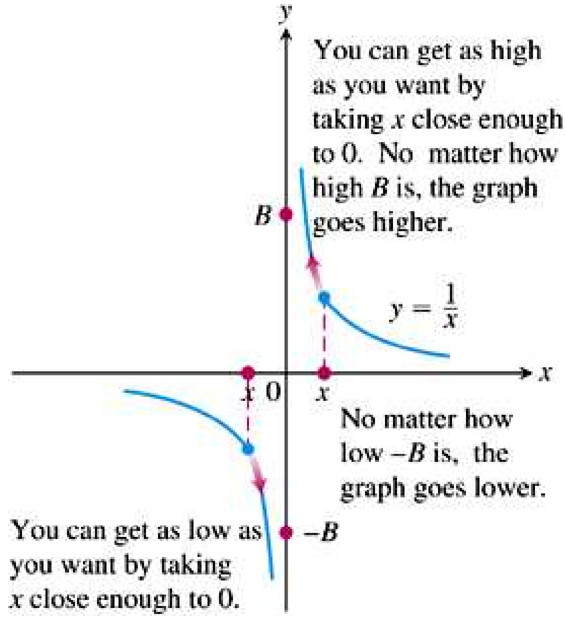
if the values of $f(x)$ decrease without bound as x approaches x_0, eventually reaching and surpassing every negative real number.

In the above example, we have

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

as the values of $1/x^2$ are positive and become arbitrarily large as x approaches 0 from the right or the left.

We can also have one sided infinite limits.
Example: $y = 1/x$



We say that $f(x)$ *approaches* ∞ *as* x *approaches* 0 *from the right* and write

$$\lim_{x\to 0^+} \frac{1}{x} = \infty\,.$$

Similarly, we say that $f(x)$ *approaches* $-\infty$ *as* x *approaches* 0 *from the left* and write

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty\,.$$

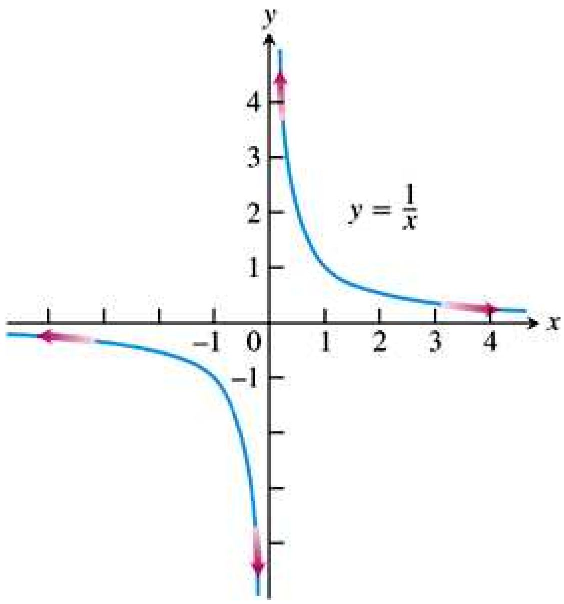
Infinite limits give rise to 'vertical asymptotes' in the graph of a function:

DEFINITION Vertical Asymptote

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

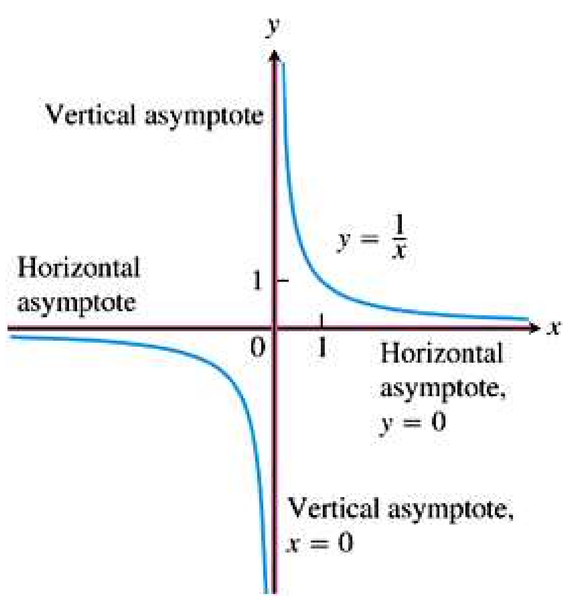
$$\lim_{x\to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x\to a^-} f(x) = \pm\infty\,.$$

Example:



Since $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$, the graph of $y = 1/x$ approaches the line $x = 0$ as x approaches 0, and this line is a vertical asymptote of the graph.

Summary The graph of $y = 1/x$ has two asymptotes: the line $y = 0$ is a horizontal asymptote and the line $x = 0$ is a vertical asymptote.



Example: Find the asymptotes of

$$f(x) = -\frac{8}{x^2 - 4}.$$

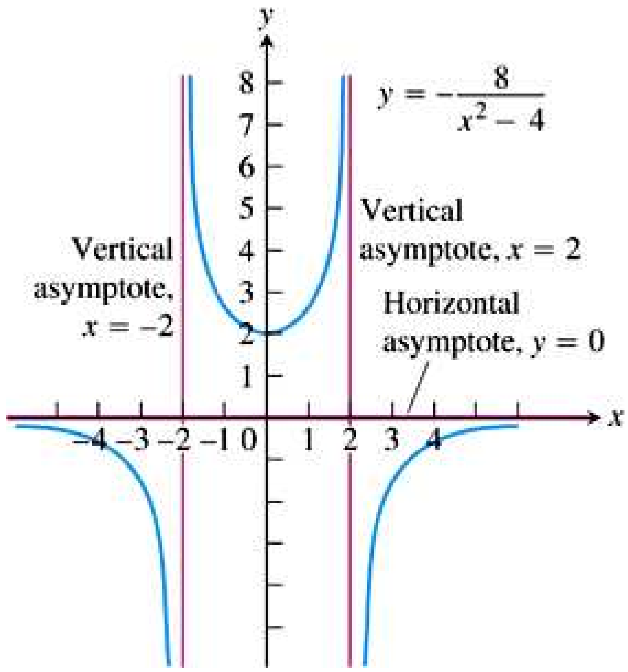
We determine the behaviour of $f(x)$ as $x \to \pm\infty$ and as $x \to \pm 2$. (Why?) We have:

(a) $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to -\infty} f(x)$;

(b) $\lim_{x \to -2^-} f(x) = -\infty$, and $\lim_{x \to -2^+} f(x) = \infty$;

(c) $\lim_{x \to 2^-} f(x) = \infty$, and $\lim_{x \to 2^+} f(x) = -\infty$ (this follows from (b) because $f(x)$ is an *even* function. Why?)

Hence the asymptotes of f are the three lines $y = 0$, $x = -2$ and $x = 2$. (Why?)



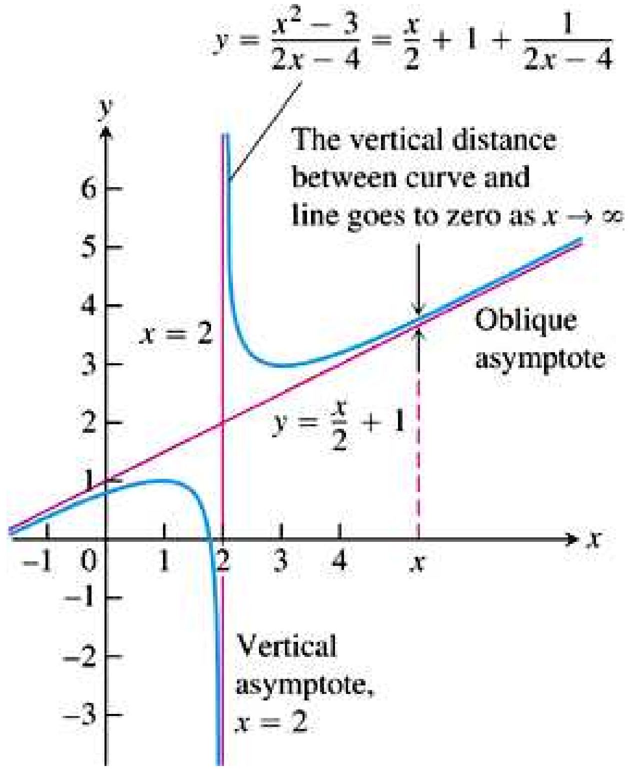
Example: Find the asymptotes of

$$f(x) = \frac{x^2 - 3}{2x - 4}.$$

Since the degree of the numerator is greater than the degree of the denominator, we first use polynomial division to rewrite the formula for $f(x)$. We have

$$f(x) = \frac{x}{2} + 1 + \frac{1}{2x - 4} \quad \text{for all } x \neq 2.$$

Hence f has one oblique asymtote given by the line $y = \frac{x}{2} + 1$ and one vertical asymptote given by the line $x = 2$.



We say that the term $\frac{x}{2} + 1$ dominates $f(x)$ when $|x|$ is large and that the term $\frac{1}{2x-4}$ dominates $f(x)$ when x is close to 2.

Continuity

Informally a function defined on an interval is continuous if we can sketch its graph in one continuous motion without lifting our pen from the paper. To give a more precise definition we first define what it means for a function to be continuous at a single point in its domain, and to do this we must distinguish between different kinds of points in the domain.
Definition Let $D \subset \mathbb{R}$ and $x \in D$. Then:

- x is an *interior point* of D if we have $x \in I$ for some open interval $I = (a, b) \subseteq D$;
- x is a *left end-point* (respectively *right end-point* of D) if x is not an interior point of D and we have $x \in I$ for some half-closed interval $I = [x, b) \subseteq D$ (respectively $I = (a, x] \subseteq D$);
- x is an *isolated point* of D if x is neither an interior point nor an end-point.

Example: Let $D = [1, 2] \cup (3, 4] \cup \{5\}$. Then D has one left end-point, 1; two right endpoints 2,4; one isolated point 5; and all other points in D are interior points.

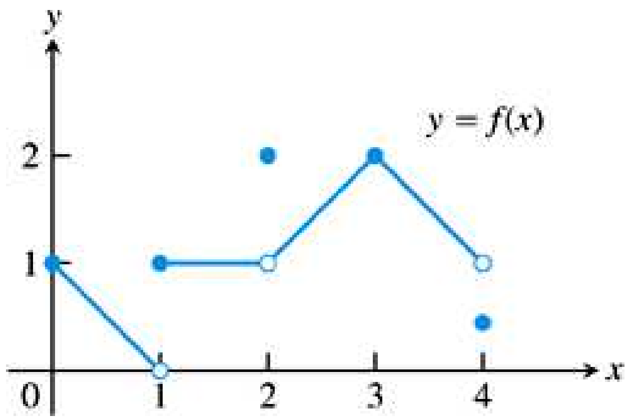
We can now define continuity at a point:

Definition Let f be a function with domain $D \subset \mathbb{R}$. Then:

- f is *continuous* at an interior point c of D if $\lim_{x\to c} f(x)$ exists and is equal to $f(c)$.
- f is *continuous* at a left end-point a of D if $\lim_{x\to a^+} f(x)$ exists and is equal to $f(a)$.

- f is *continuous* at a right end-point b of D if $\lim_{x \to b^-} f(x)$ exists and is equal to $f(b)$.
- f is *continuous* at every isolated point of D.[1]

Example: $f : [0, 4] \to \mathbb{R}$



The function f is continuous at all points in $[0, 4]$ *except at* $x = 1, x = 2$ and $x = 4$.

[1]In this module our domains will never have isolated points so this part of the definition will never be used.