Queen Mary University of London

MTH4100 Calculus I

Lecture notes for Week 3

Thomas' Calculus, Sections 1.3, 2.1 and 2.2

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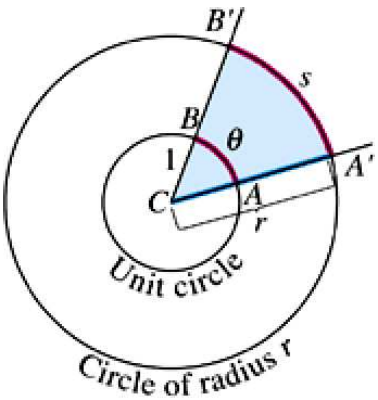
Trigonometric functions

The **radian measure** of the angle ACB is the length θ of the arc AB on the unit circle.



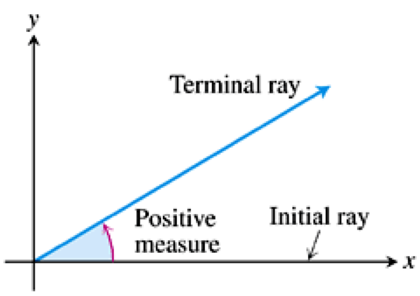
Thus $s = r\theta$ is the length of the arc on a circle of radius r when the angle it subtends θ is measured in radians.

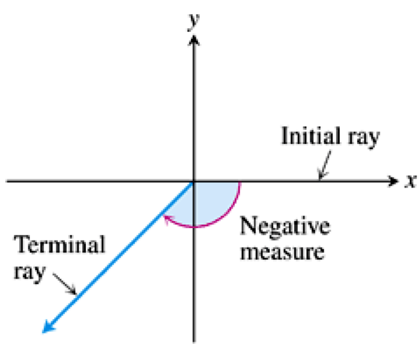
Conversion formula between degrees and radians: 360° corresponds to 2π, hence:

$$\boxed{\frac{\text{angle in radians}}{\text{angle in degrees}} = \frac{\pi}{180}}$$

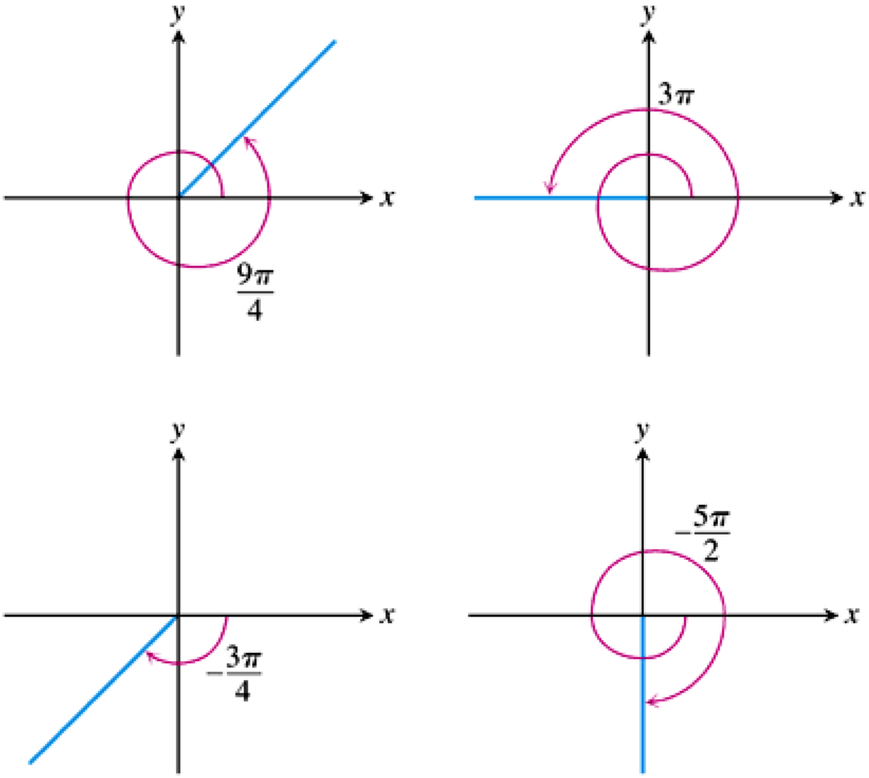
Note that:

- angles are **oriented**;
- *positive angle* are measured counter-clockwise;
- *negative angle* are measured clockwise.

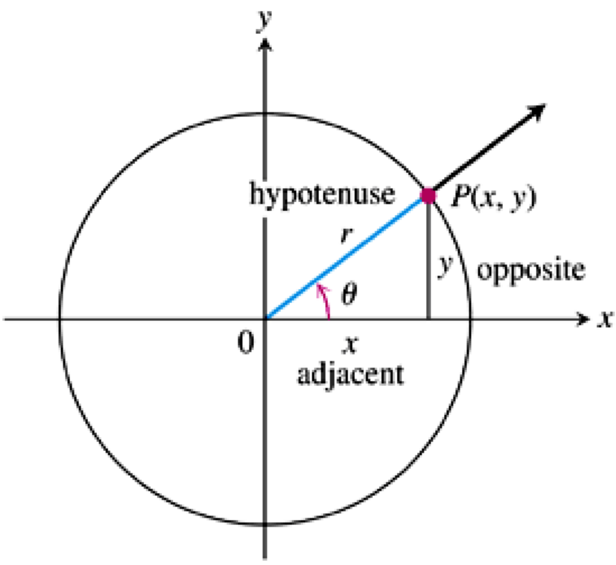




Note that angles can be *larger* (counter-clockwise) or *smaller* (clockwise) than 2π:



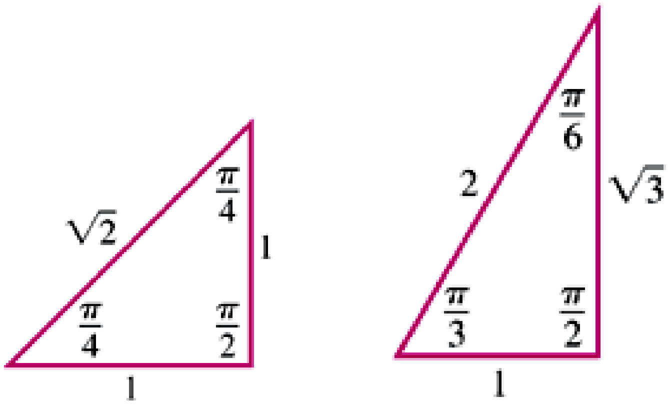
The six basic trigonometric functions:



$$
\begin{array}{rlrl}
\textbf{sine:} & \sin\theta = \frac{y}{r} & \textbf{cosecant:} & \csc\theta = \frac{r}{y} \\
\textbf{cosine:} & \cos\theta = \frac{x}{r} & \textbf{secant:} & \sec\theta = \frac{r}{x} \\
\textbf{tangent:} & \tan\theta = \frac{y}{x} & \textbf{cotangent:} & \cot\theta = \frac{x}{y}
\end{array}
$$

Note that these definitions hold not just for $0 \leq \theta \leq \pi/2$ but for all $\infty < \theta < \infty$.

It is useful to memorize the following two special triangle because *exact values* of the trigonometric functions can be read from them.

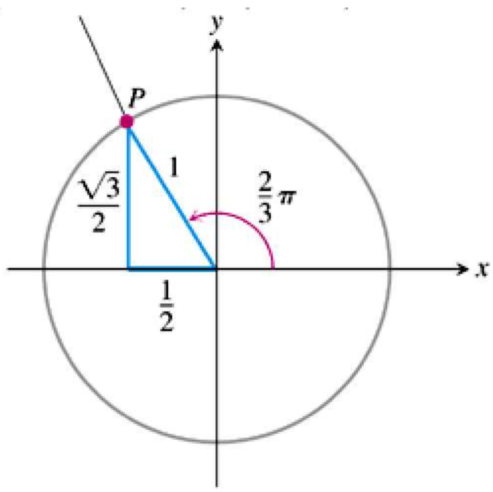


Examples:

(a)

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad ; \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

(b)



$$\sin\left(\frac{2}{3}\pi\right) = \frac{y}{r} = \frac{\sqrt{3}}{2} \qquad \csc\left(\frac{2}{3}\pi\right) = \frac{r}{y} = \frac{2}{\sqrt{3}}$$

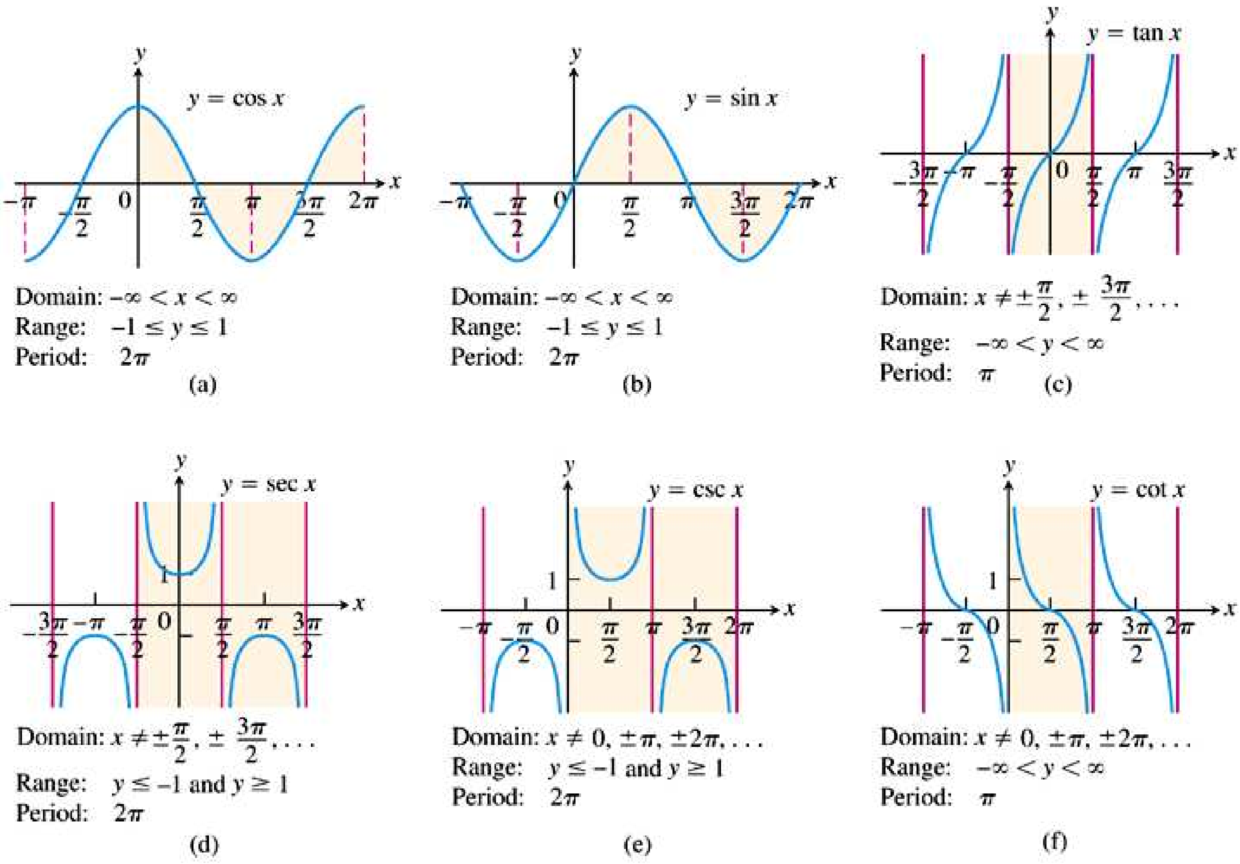
$$\cos\left(\frac{2}{3}\pi\right) = \frac{x}{r} = -\frac{1}{2} \qquad \sec\left(\tfrac{2}{3}\pi\right) = \tfrac{r}{x} = -2$$

$$\tan\left(\frac{2}{3}\pi\right) = \frac{y}{x} = -\sqrt{3} \qquad \cot\left(\frac{2}{3}\pi\right) = \frac{x}{y} = -\frac{1}{\sqrt{3}}$$

DEFINITION Periodic Function

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x+p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

Since for any angle $\theta \in \mathbb{R}$, all six trigonometric functions will take the same value at θ and $\theta + 2\pi$ (why?) all six trigonometric functions are periodic. We can determine their periods by considering their graphs:



Some important trigonometric identities

We first state a result which extends Pythagorus' Theorem.

Theorem 1 (The law of the cosine) *Suppose a, b, c are the lengths of three sides of a triange and the angle between the sides of lengths a, b is θ. Then*

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Proof See Thomas' Calculus, p. 26-27.

We can use this result to obtain our first trigonometric identity.[1]

Theorem 2 *Suppose $\theta, \phi \in \mathbb{R}$. Then*

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi \tag{1}$$

Proof See Thomas' Calculus, p.29, Exercise 57.

We can use (1) to deduce many other trigonometric identities. For example

- If we take $\phi = \theta$ in (1) we obtain
$$1 = \cos^2\theta + \sin^2\theta.$$
- If we take $\phi = -\theta$ in (1) we obtain
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta.$$
- If we take $\theta = \pi/2 - A$ and $\phi = B$ in (1) we obtain
$$\sin(A + B) = \sin A\cos B + \cos A\sin B.$$

Reading Assignment: **Read Thomas' Calculus**

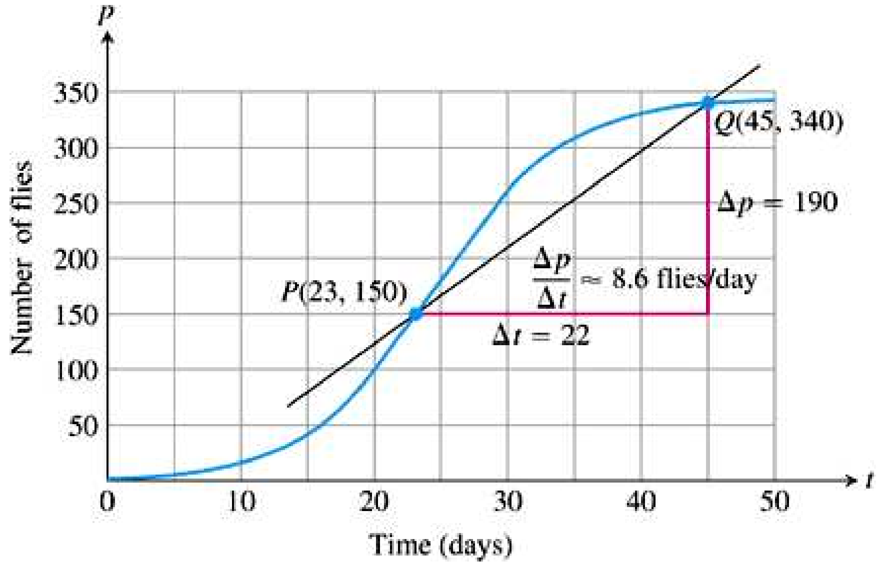
- short **paragraph** about ellipses, p. 18/19
- **Section 1.3**, p. 25-28 about trigonometric function symmetries and identities

You will need this for Coursework 2.

[1] An identity is an equation which is valid for *all* values of the variable(s) it contains. The equation $\cos\theta = 1$ is *not* an identity, because it is only true for *some* values of θ, not all.

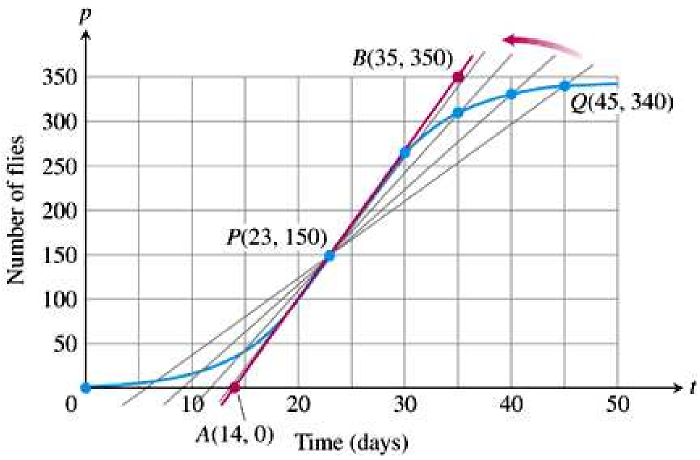
Rates of change and limits

Example: growth of a fruit fly population measured experimentally. It is straightforward to calculate the average rate of change from day 23 to day 45.



We can also calculate the *instantaneous rate of change* at a particular time on a specific day, e.g. 00:00 on day 23, by finding the average rates of change over *increasingly short time intervals* starting at time 00:00 on day 23:

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\frac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\frac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\frac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\frac{265 - 150}{30 - 23} \approx 16.4$



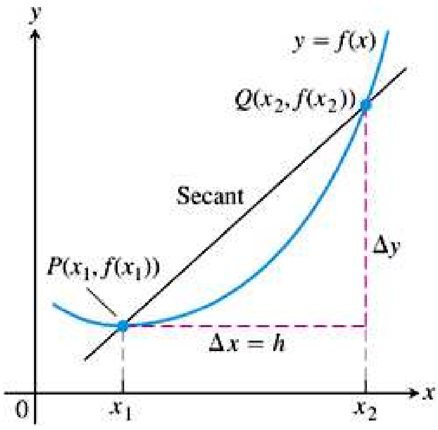
The lines PQ approach the red *tangent* AB at the point P with slope

$$\frac{350 - 0}{35 - 14} \simeq 16.7 \text{ flies/day}$$

Definition The *average rate of change* of a function f over an interval $I = [x_1, x_2]$ is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h} \tag{2}$$

where $h = x_2 - x_1 \neq 0$.



To move from average rates of change to instantaneous rates of change we need to consider 'the limiting value' of (2) as h approaches zero.

Informal Definition Let f be a function defined everywhere in an open interval containing x_0 (except possibly at x_0 itself). If $f(x)$ gets 'arbitrarily close' to a number L for all x 'sufficiently close to' but not equal to x_0, then we say that f *approaches the* limit L *as* x *approaches* x_0, and we write

$$\lim_{x \to x_0} f(x) = L \, .$$

This is read as "the limit of $f(x)$ as x approaches x_0 is equal to L."[2]

Example: How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave as x approaches 1?

We can *simplify* this formula for $f(x)$ when $x \neq 1$. We have:

$$f(x) = \frac{(x-1)(x+1)}{x - 1} = x + 1 \text{ for } x \neq 1 \, .$$

This *suggests* that

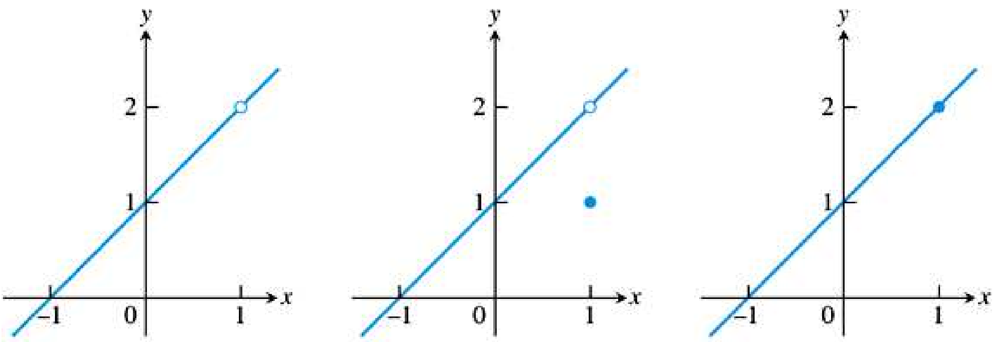
$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (x + 1) = 1 + 1 = 2,$$

see graph (a) below.

[2] This definition is 'informal' because the terms "arbitrarily close to" and "sufficiently close to" are not precise. It will serve our purpose for this module (which is to get an intuitive understanding of limits). But it is still worthwhile to compare the informal definition with the precise definition below.

Definition Let f be a function defined everywhere in an open interval containing x_0 (except possibly at x_0 itself). Then we say that f *approaches the* limit L *as* x *approaches* x_0 if, for all real numbers $a > 0$, we can choose a real number $b > 0$ such that we have $|f(x) - L| < a$ whenever $0 < |x - x_0| < b$.

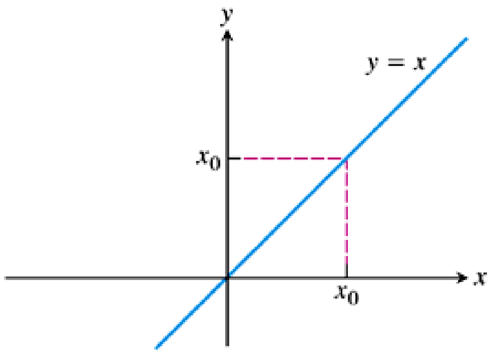
This definition will be used next year in MTH5104 Convergence and Continuity. If you want to find out more now you should read Thomas' Calculus, Section 2.3.



(a) $f(x) = \dfrac{x^2-1}{x-1}$ (b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

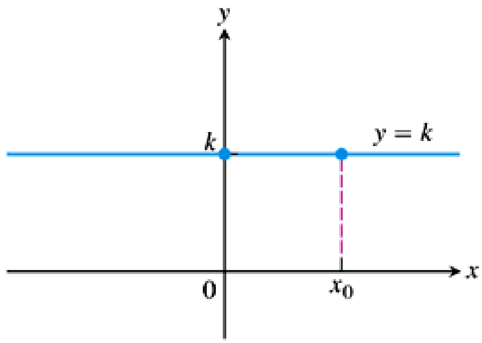
Note: The limit of a function at a point x_0 does not depend on the value the function takes at x_0. The function f in the above example is not defined at $x = 1$. But if we define a new function by choosing an arbitrary value for $f(1)$, then the new function will still have the same limit at $x = 1$ as f, see graphs (b) and (c) above. Note, however, that only the function h shown in (c) has the property that the limit value and the function value at $x = 1$ are the same i.e. $\lim_{x\to 1} h(x) = h(1)$.

Some functions can have limits at all point on the real line. For example:
(a) $f(x) = x$.



For any $x_0 \in \mathbb{R}$ we have $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} x = x_0$. For example $\lim\limits_{x\to 3} x = 3$.

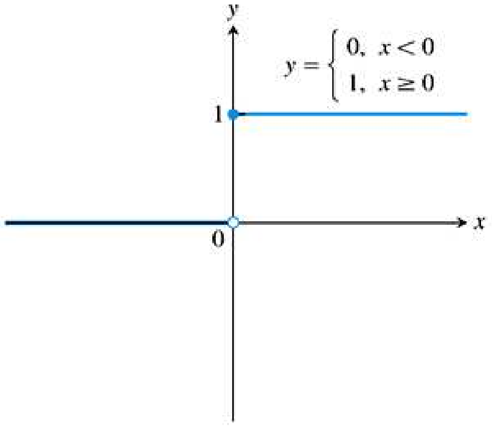
(b) $f(x) = k$ for some constant $k \in \mathbb{R}$.



For any $x_0 \in \mathbb{R}$ we have $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} k = k$. For example, when $k = 5$, we have $\lim\limits_{x\to -12} 5 = \lim\limits_{x\to 7} 5 = 5$.

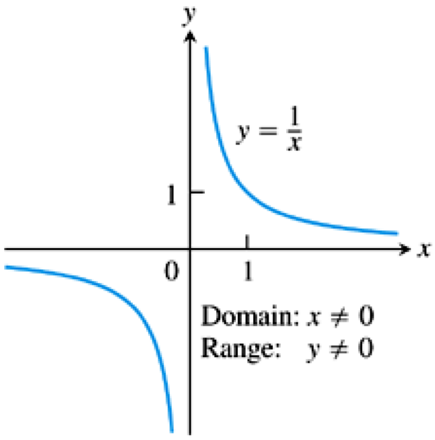
For other functions, limits can fail to exist at some points. For example, the following functions have no limit at $x = 0$.

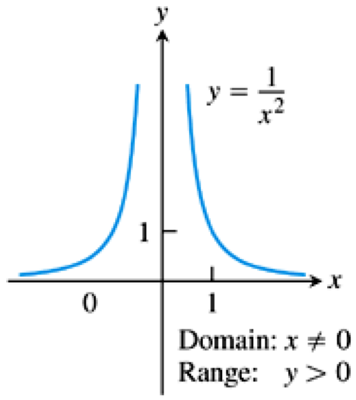
(a)



limit fails to exist because the function *jumps* as we approach $x = 0$.

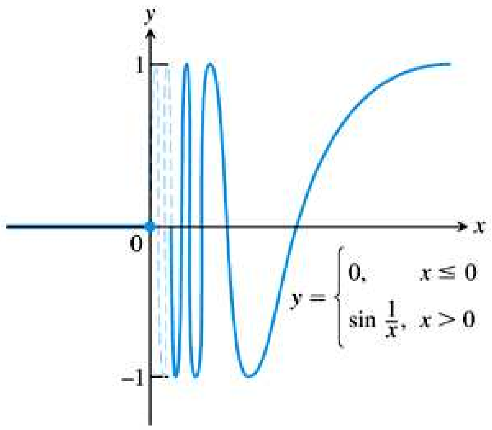
(b)





limit fails to exist because the function *becomes too large* as we approach $x = 0$.

(c)



limit fails to exist because the function *oscillate too much* as we approach $x = 0$.

We have seen that for any $c \in \mathbb{R}$ and any constant k:

$$\lim_{x \to c} x = c$$

and

$$\lim_{x \to c} k = k.$$

The following theorem allows us to use these two results to calculate limits of functions that are algebraic combinations of the above two functions (such as polynomial and rational functions).

Theorem 3 (Limit laws) *Suppose that c, L, M are real numbers, and f and g are functions such that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then*

1. Sum Rule: $\lim_{x\to c}(f(x) + g(x)) = L + M$
   The limit of the sum of two functions is the sum of their limits.
2. Difference Rule: $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. Constant Multiple Rule: $\lim_{x\to c}(kf(x)) = kL$ *for any constant $k \in \mathbb{R}$.*
4. Product Rule: $\lim_{x\to c}(f(x)g(x)) = LM$
5. Quotient Rule: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \neq 0$
6. Power Rule: $\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$ *for any integers r, s such that $L^{r/s}$ is a real number.*

To prove this theorem we need to use the precise definition of a limit, see Thomas' Calculus Section 2.3 and Appendix 2, or MTH5104. (We cannot prove anything using our intuitive definition of a limit!)

Examples: Find $\lim_{x\to 3} f(x)$ when

- $f(x) = x^3 - 4x + 2$. We have

$$\begin{aligned} \lim_{x\to 3}(x^3 - 4x + 2) &= \lim_{x\to 3} x^3 - \lim_{x\to 3} 4x + \lim_{x\to 3} 2 \qquad \text{[by Theorem 3(1,2)]} \\ &= 3^3 - 4 \cdot 3 + 2 \qquad \text{[by Theorem 3(3)]} \\ &= 17 \end{aligned}$$

- $f(x) = \sqrt{4x^2 - 3}$. We have $\lim_{x\to 3}(4x^2 - 3) = 4 \cdot 3^2 - 3 = 33$ as in the previous example. Hence

$$\lim_{x\to 3} \sqrt{4x^2 - 3} = \sqrt{33} \qquad \text{[by Theorem 3(6)]}$$

A similar argument shows that we can find limits of *any* polynomial function just by 'substituting the value of x'.

THEOREM 2 **Limits of Polynomials Can Be Found by Substitution**
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

More generally we can find limits of *any* rational functions by 'substituting the value of x' as long as the denominator does not become zero.

THEOREM 3 **Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero**
If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example:

$$\lim_{x \to 1} \frac{x^2 + x - 1}{x^2 - 2x} = \frac{1^2 + 1 - 1}{1^2 - 2} = -1.$$

Sometimes the numerator and denominator of a rational function can *both* become zero when we substitute a value of x. If this happens, we can try to first use an 'algebraic simplification' which gives us a non-zero denominator, and then calculate the limit by substitution.
Examples:

- Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

When we substitute $x = 1$ the numerator and denominator both become zero. But an algebraic simplification is possible:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x+2)(x-1)}{x(x-1)} = \frac{x+2}{x} \text{ when } x \neq 1.$$

Therefore,

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x+2}{x} = 3.$$

- Evaluate

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}.$$

Algebraic simplification

$$\begin{aligned}
\frac{\sqrt{x^2 + 100} - 10}{x^2} &= \frac{\sqrt{x^2 + 100} - 10}{x^2} \times \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10} \\
&= \frac{(x^2 + 100) - 100}{x^2(\sqrt{x^2 + 100} + 10)} \\
&= \frac{1}{\sqrt{x^2 + 100} + 10}
\end{aligned}$$

when $x \neq 0$. Therefore

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10} = \frac{1}{20}$$