MTH4100 Calculus I

Lecture notes for Week 2

Thomas' Calculus, Sections 1.1 and 1.2

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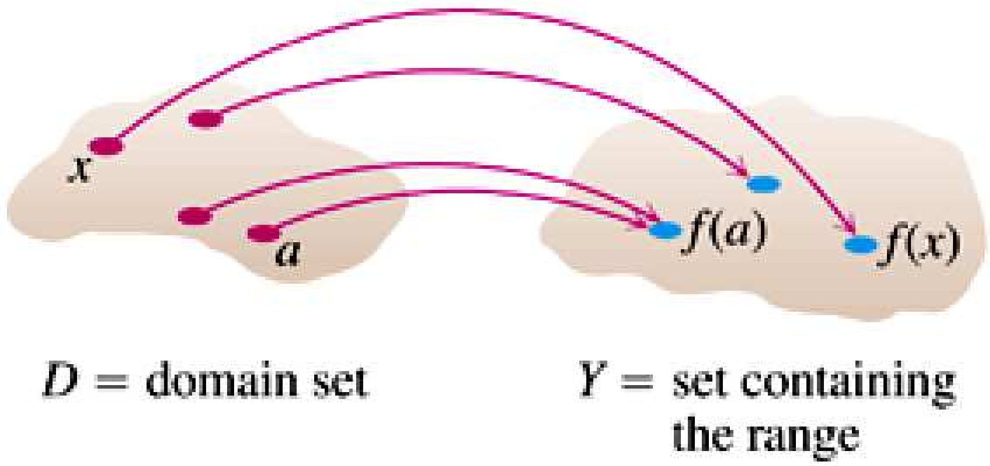
What is a function?

Definition A *function f from a set D to a set Y* is a rule that assigns an element $f(x)$ of Y to each element x of D.

Note that functions have a *uniqueness* property - there is only one value $f(x) \in Y$ assigned to each $x \in D$.



- The set D of all possible input values is called the *domain* of f.
- The set Y which contains all possible output values is called the *codomain* of f.
- The set R consisting of all possible *output values* of $f(x)$ as x varies throughout D is called the *range* of f.[1]
- We write f maps D to Y symbolically as $f : D \to Y$.
- We write *f maps x to $f(x)$* symbolically as $f : x \mapsto f(x)$.

Note that different arrow symbols $\to$ and $\mapsto$ are used in each case.

We often think of the input and output values of a function as *variables*. The function tells us how to determine the value of the output variable y from the value of the input variable x. We write $y = f(x)$ and refer to x as the *independent variable* and y as the *dependent variable*. The function f acts like a "black box" which inputs x and outputs $y = f(x)$.

[1]Note that $R \subseteq Y$ i.e. the range is contained in (but not necessarily equal to) the codomain.

Examples:
y is the height of the floor of the lecture hall depending on the distance x from the white-board;
y is the stock market index depending on the time x;
y is the volume of a sphere depending on its radius x.

In general the domain D and the codomain Y of a function f can be any sets. In this module, however, *we will always take D and Y to be subsets of $\mathbb{R}$*. In addition we will often be lazy and not specify the domain and codomain of f explicitly: in this case we will assume that *the domain of f is the the largest set of real numbers for which the definition of f makes sense* and that *the codomain of f is $\mathbb{R}$*.
Examples:

Function	**Domain**	**Codomain**	**Range**
$y = x^2$	$(-\infty, \infty)$	$\mathbb{R}$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$\mathbb{R}$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$\mathbb{R}$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$\mathbb{R}$	$[0, 1]$

Remark: A function is fully specified by not only giving the rule f, but also giving its domain D, and its codomain Y. Thus

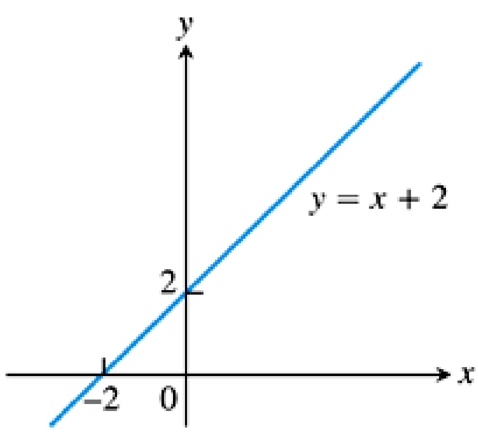
$$f : \mathbb{R} \to \mathbb{R} \text{ defined by } f : x \mapsto x^2$$

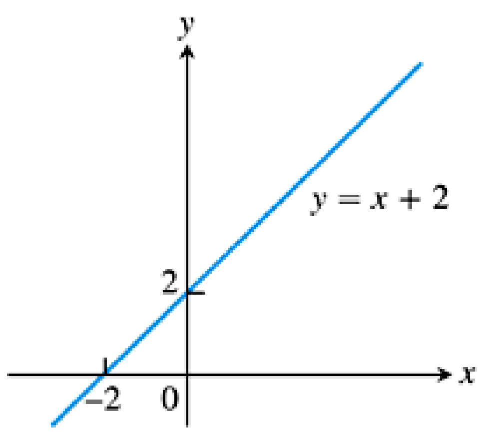
and

$$g : [0, \infty) \to \mathbb{R} \text{ defined by } g : x \mapsto x^2$$

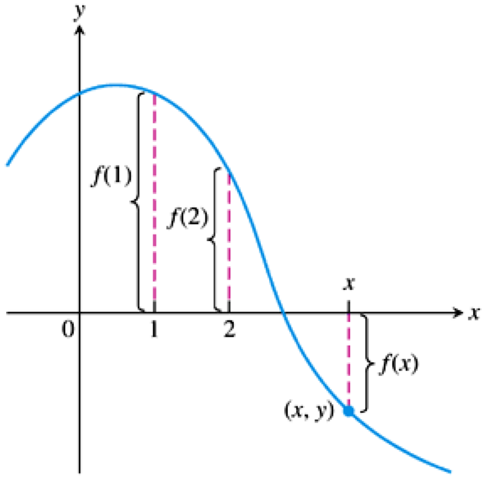
are *different* functions since they have different domains.

Definition The *graph* of a function $f : D \to \mathbb{R}$ is of the set of all points $(x, f(x))$ in the plane whose coordinates are the input-output pairs for f.
Example:





Given a function f, we can *sketch* its graph by plotting some of its points $(x, f(x))$ in the plane and then 'joining them up'. Calculus will help us do this more accurately.



The 'y-coordinate' is the *height* of the point $(x, f(x))$ above x.

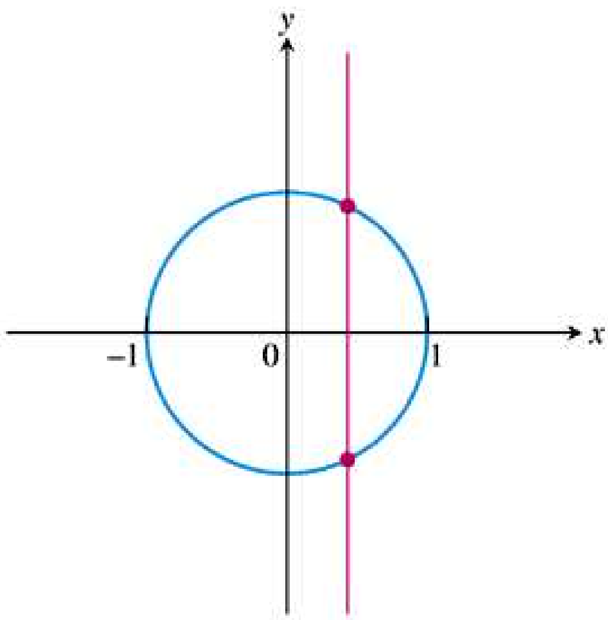
Definition A *curve* is of the set of all points (x, y) in the cartesian plane whose coordinates satisfy some equation involving the variables x, y.

The graph of a function f is a special kind of curve since it is defined by the equation $y = f(x)$. However some curves are not graphs of any function. To see this we use the following observation.

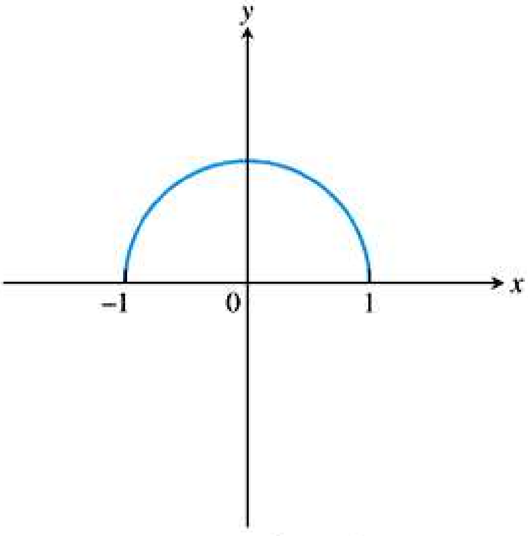
Recall that a function f can have only *one value $f(x)$ assigned to each x* in its domain. This leads to *the vertical line test*:

No vertical line can intersect the graph of a function *more than once.*

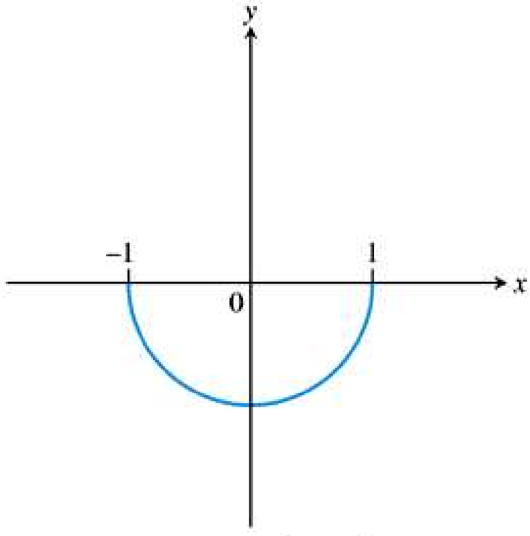
Example



(a) $x^2 + y^2 = 1$



(b) $y = \sqrt{1 - x^2}$



(c) $y = -\sqrt{1 - x^2}$

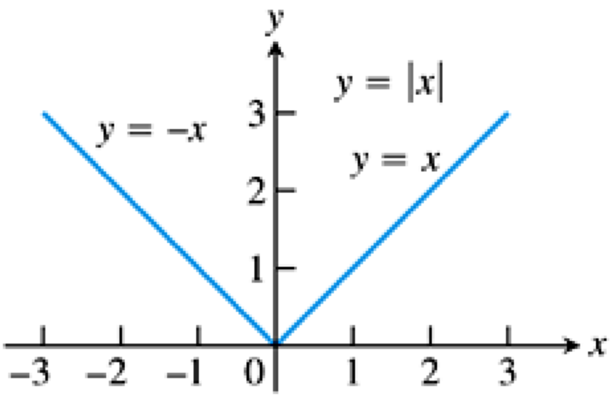
The curve shown in (a) is *not* the graph of a function since it fails the vertical line test. The curves in (b) and (c) *are* graphs of functions.

Definition A *piecewise defined function* is a function that is described by using different formulas on different parts of its domain.

Examples:

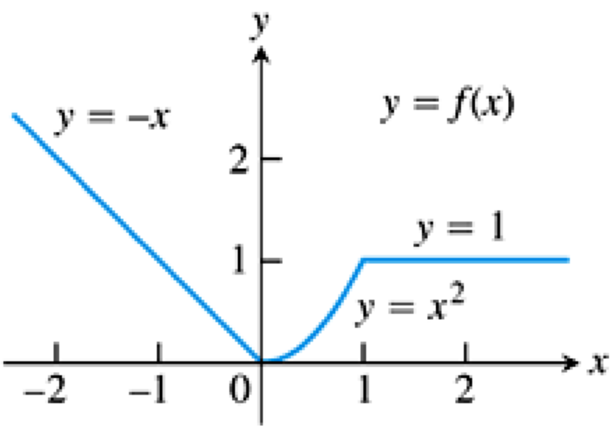
- the *absolute value function*

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



- another piecewise defined function

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

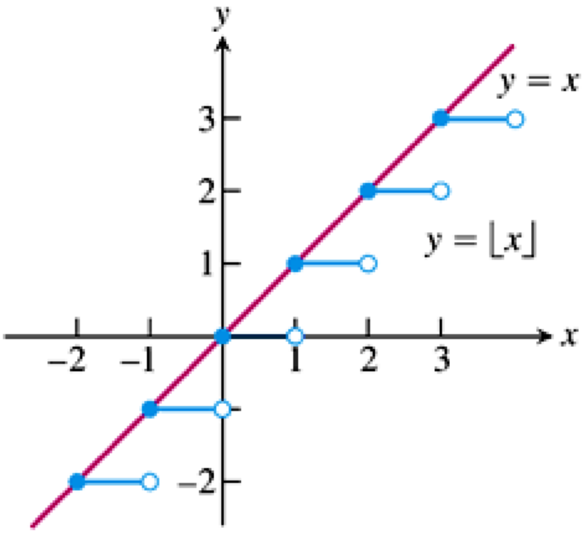


- the *floor function*

$$f(x) = \lfloor x \rfloor$$

is defined by taking $\lfloor x \rfloor$ to be the greatest integer which is less than or equal to x. Thus

$$\lfloor 1.3 \rfloor = 1, \ \lfloor -2.7 \rfloor = -3$$

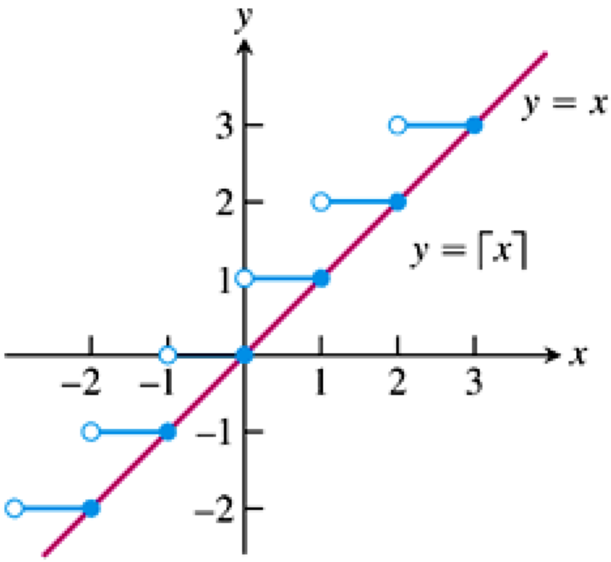


- the *ceiling function*

$$f(x) = \lceil x \rceil$$

is defined by taking $\lceil x \rceil$ to be the smallest integer which is greater than or equal to x. Thus

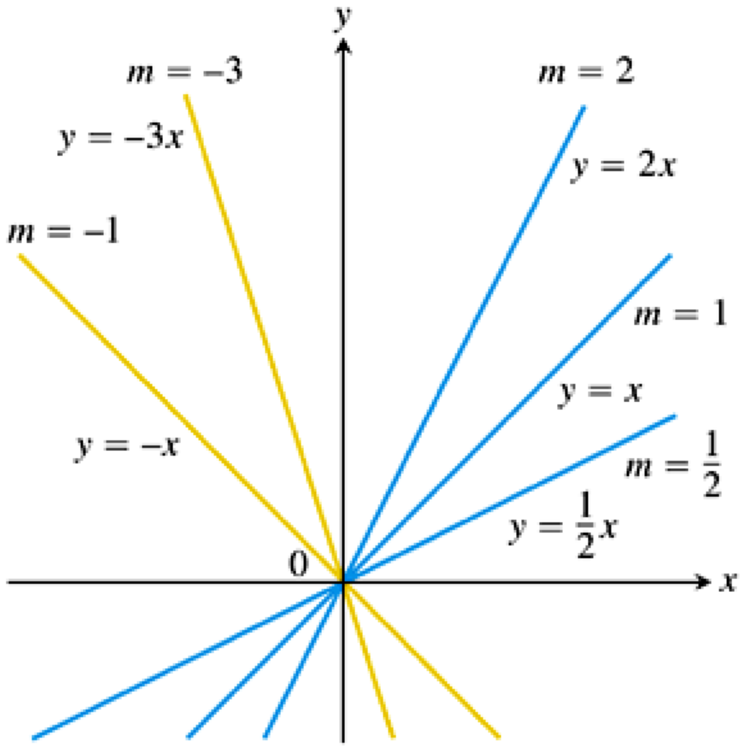
$$\lceil 3.5 \rceil = 4, \ \lceil -1.8 \rceil = -1$$



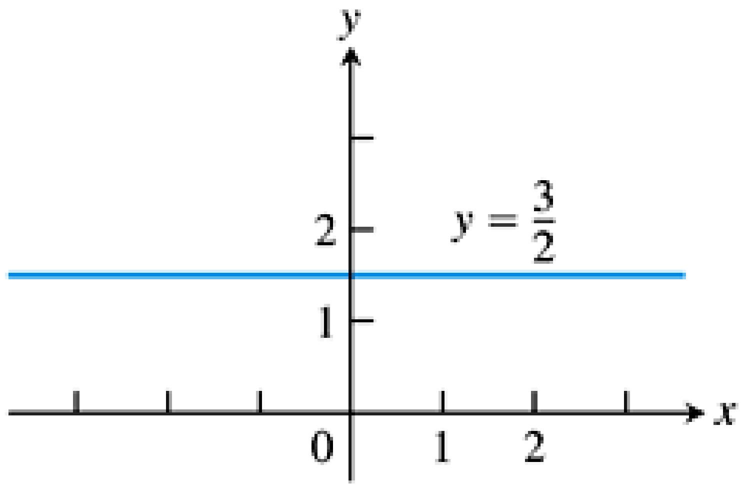
Some important functions

- **linear function:** $\boxed{f(x) = mx + b}$ for some $m, b \in \mathbb{R}$

When $b = 0$, $f(x) = mx$ and the graph of f is a line through the origin.

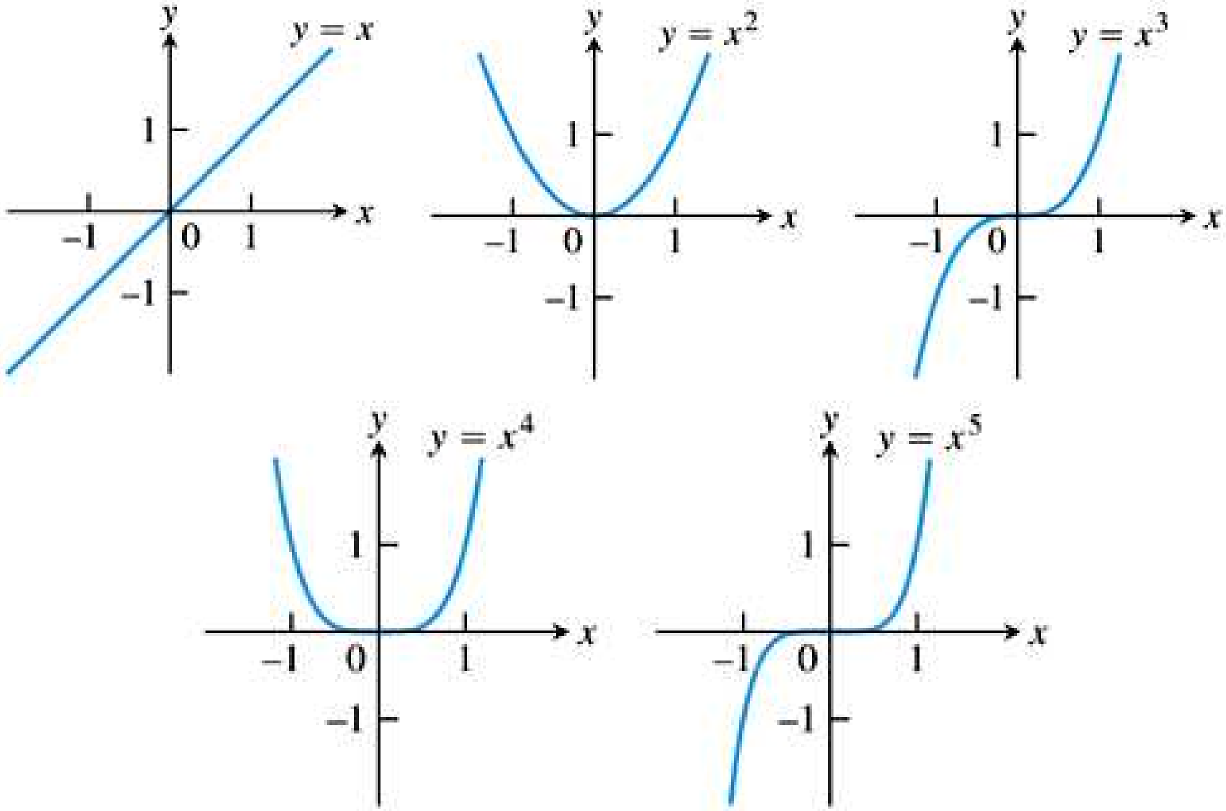


When $m = 0$, $f(x) = b$ and f is a *constant function*.

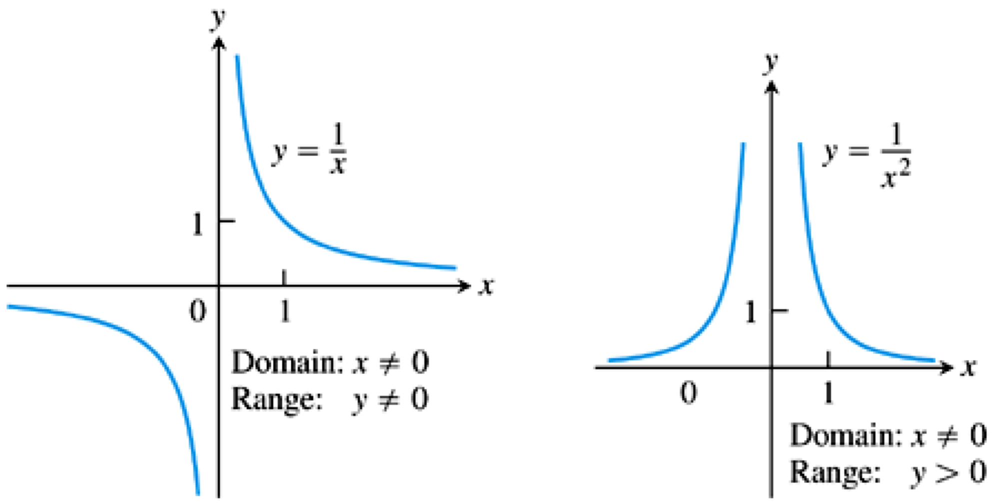


• **power function:** $\boxed{f(x) = x^a}$ for $a \in \mathbb{R}$.

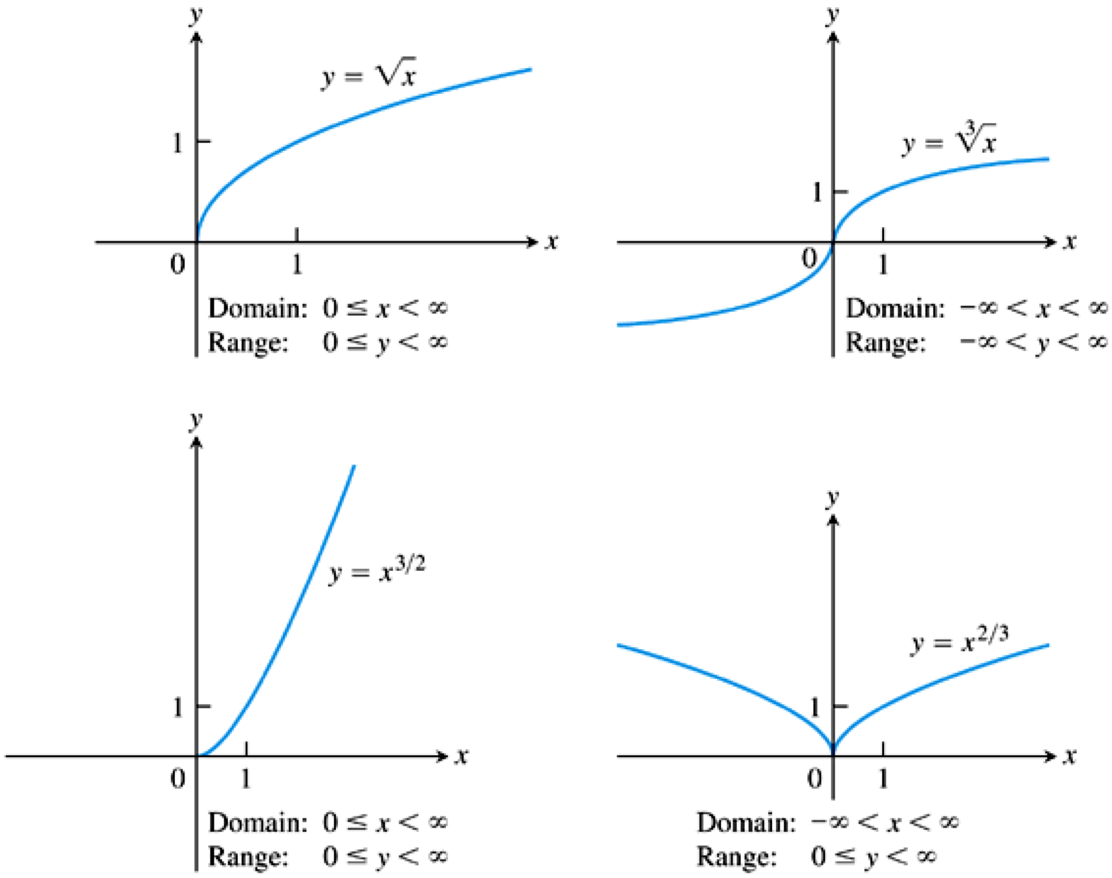
Graphs of $f(x) = x^a$ for $a = 1, 2, 3, 4, 5$



Graphs of $f(x) = x^a$ for $a = -1, -2$



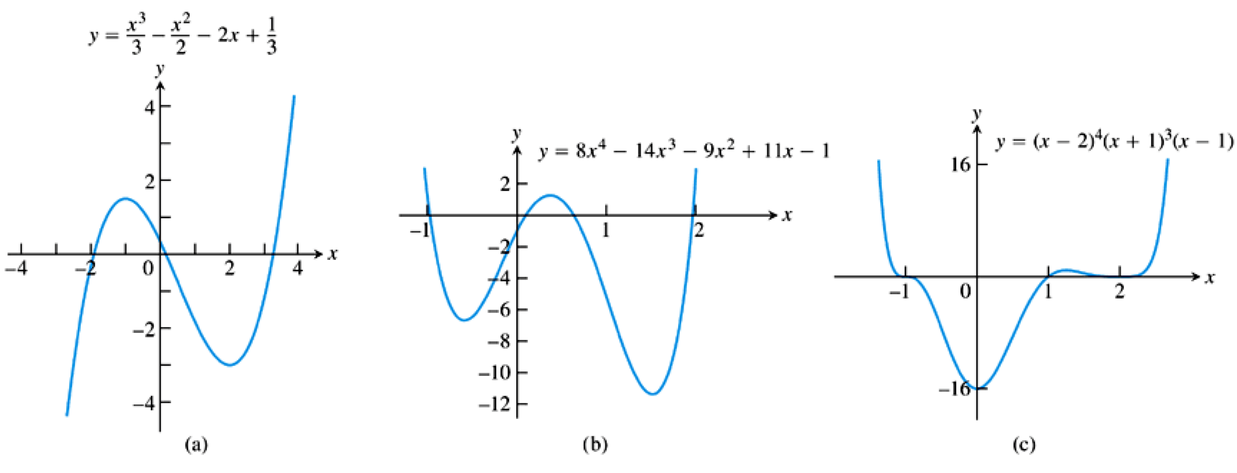
Graphs of $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}$



• **polynomial function:** $\boxed{p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}$ for $n \in \mathbb{Z}$ with $n \geq 0$, and $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{R}$ with $a_n \neq 0$.

We say that: $p(x)$ is a *polynomial in* x; $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{R}$ are the *coefficients* of $p(x)$; n is the *degree* of $p(x)$. Constant functions correspond to polynomials of degree zero. Linear functions $f(x) = mx + b$ with $m \neq 0$ correspond to polynomials of degree one.

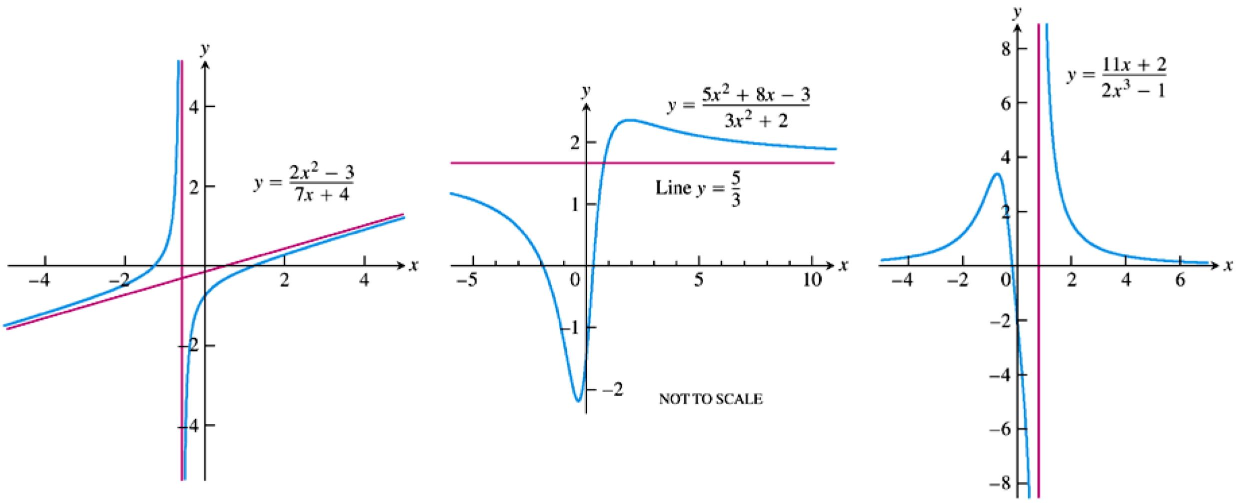
Three polynomial functions and their graphs



• **rational functions:** $\boxed{f(x) = \dfrac{p(x)}{q(x)}}$ where $p(x)$ and $q(x)$ are polynomials.

Note that the domain of f is $\{x \in \mathbb{R} : q(x) \neq 0\}$ since we can never divide by zero.

Three rational functions and their graphs

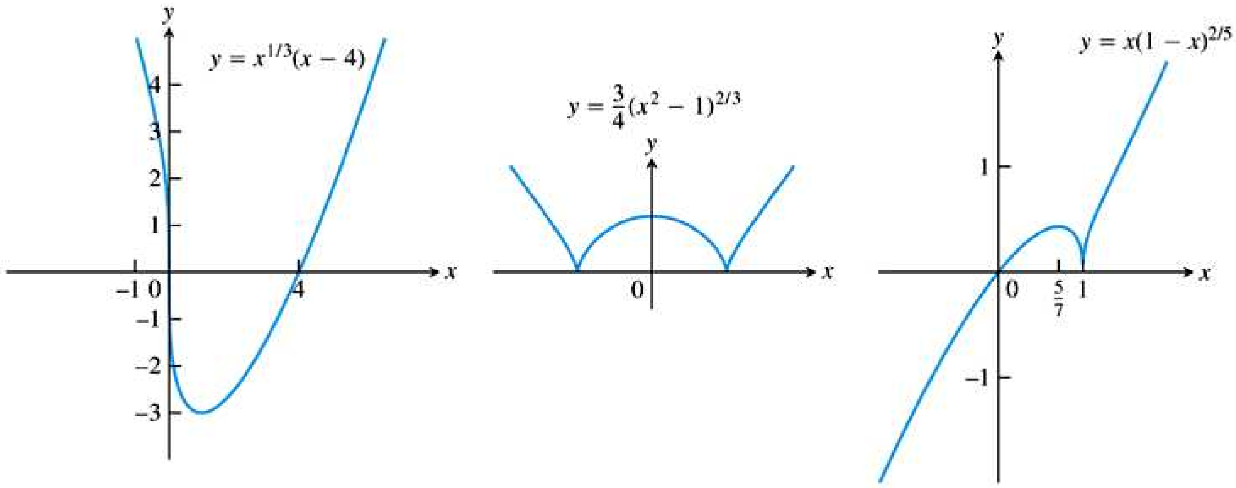


(a) (b) (c)

We will see many other types of functions later in this module. For example:

algebraic functions: any function constructed from polynomials using algebraic operations (including taking roots)

examples:



(a) (b) (c)

trigonometric functions

exponential and **logarithmic functions**

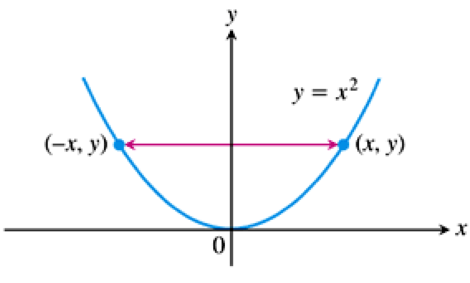
Special types of functions

Definition A function $f : D \to \mathbb{R}$ is *increasing* on some interval $I \subseteq D$ if $f(x_1) \leq f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 \leq x_2$. (Informally, f is increasing if the graph of f "climbs" or "rises" as we move along I from left to right.)
Similarly f is *decreasing* on I if $f(x_1) \geq f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 \leq x_2$. (Informally, f is decreasing if the graph of f "descends" or "falls" as we move along I from left to right.)

Examples:

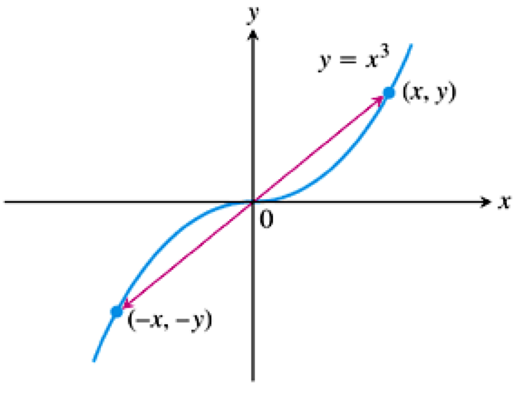
function	where increasing	where decreasing
$y = x^2$	$0 \leq x < \infty$	$-\infty < x \leq 0$
$y = 1/x$	nowhere	$-\infty < x < 0$ and $0 < x < \infty$
$y = 1/x^2$	$-\infty < x < 0$	$0 < x < \infty$
$y = x^{2/3}$	$0 \leq x < \infty$	$-\infty < x \leq 0$

Definition A function $f : \mathbb{R} \to \mathbb{R}$ is *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. (This is the same as saying its graph is symmetric about the y-axis.)
Similarly, f is *odd* if $f(-x) = -f(x)$ for $x \in \mathbb{R}$. (This is the same as saying its graph is symmetric about the origin.)
Examples:
(a) $f(x) = x^2$



(a)

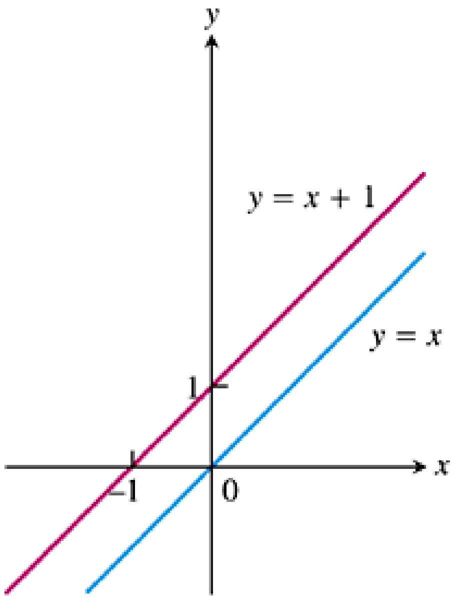
$f(-x) = (-x)^2 = x^2 = f(x)$ so f is an even function; its graph is *symmetric about the y-axis.*
(b) $f(x) = x^3$



(b)

$f(-x) = (-x)^3 = -x^3 = -f(x)$: odd function; its graph is symmetric about the origin.

(c) $f(x) = x$ and $g(x) = x + 1$



$f(-x) = -x = -f(x)$ so f is an odd function
$g(-x) = -x + 1 \neq g(x)$ and $-g(x) = -x - 1 \neq g(-x)$ so g is neither even nor odd.

Combining functions

Algebraic Combinations

Suppose $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are functions. Then we can define new functions $f + g$, $f - g$ and fg with domain $D \cap E$ as follows:

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$

We can also define the function f/g with domain $\{x \in D \cap E \ : \ g(x) \neq 0\}$ by:

$$(f/g)(x) = f(x)/g(x)$$

We refer to these new functions as the *sum, difference, product, and quotient* of f and g. A special case of the product is when we multiply a function g by a constant $c \in \mathbb{R}$: we obtain a new function cg where $(cg)(x) = c\,g(x)$ by taking f to be the constant function $f(x) = c$ in the above definition of product.

Examples:

$$f(x) = \sqrt{x} \quad \text{domain } D = [0, \infty)$$

$$g(x) = \sqrt{1 - x} \qquad \text{domain } E = (-\infty, 1]$$

intersection of both domains:

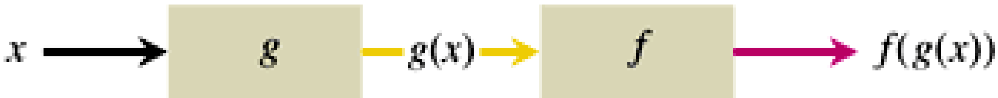
$$D \cap E = [0, \infty) \cap (-\infty, 1] = [0, 1]$$

function	formula	domain
$f+g$	$(f+g)(x)=\sqrt{x}+\sqrt{1-x}$	$[0,1]$
$f-g$	$(f-g)(x)=\sqrt{x}-\sqrt{1-x}$	$[0,1]$
$g-f$	$(g-f)(x)=\sqrt{1-x}-\sqrt{x}$	$[0,1]$
fg	$(fg)(x)=f(x)g(x)=\sqrt{x(1-x)}$	$[0,1]$
f/g	$\frac{f}{g}(x)=\frac{f(x)}{g(x)}=\sqrt{\frac{x}{1-x}}$	$[0,1)$ ($x=1$ excluded)
g/f	$\frac{g}{f}(x)=\frac{g(x)}{f(x)}=\sqrt{\frac{1-x}{x}}$	$(0,1]$ ($x=0$ excluded)

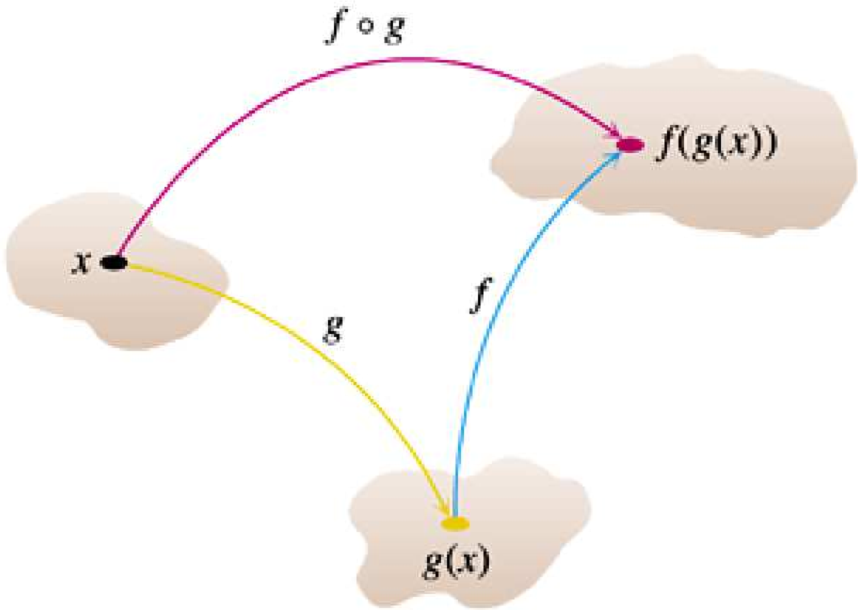
Definition Suppose $f : D \to \mathbb{R}$ and $g : E \to R$ are functions. Then the *composite* function $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x)).$$

(We read $f \circ g$ as "f composed with g". We also refer to $f \circ g$ as "the composition of f with g.")



The *domain* of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f, i.e. $\{x \in \mathbb{R} : x \in E \text{ and } g(x) \in D\}$.



Examples: (a) Suppose

$$\begin{array}{llllll} f(x) & = \sqrt{x} & \text{domain} & D = [0,\infty) & \text{range} & R = [0,\infty) \\ g(x) & = x+1 & \text{domain} & E = (-\infty,\infty) & \text{range} & S = (-\infty,\infty) \end{array}$$

Then

composite	**domain**
$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1,\infty)$
$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0,\infty)$
$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0,\infty)$
$(g \circ g)(x) = g(g(x)) = g(x) + 1 = x + 2$	$(-\infty,\infty)$

(b) Suppose

$$\begin{array}{llllll} f(x) & = \sqrt{x} & \text{domain} & D = [0,\infty) & \text{range} & R = [0,\infty) \\ g(x) & = x^2 & \text{domain} & E = (-\infty,\infty) & \text{range} & S = [0,\infty) \end{array}$$

Then

composite	**domain**
$(f \circ g)(x) = \lvert x \rvert$	$(-\infty,\infty)$
$(g \circ f)(x) = x$	$[0,\infty)$

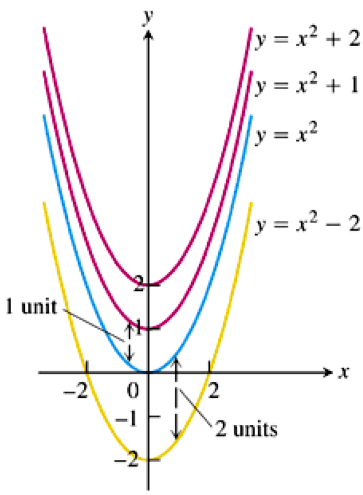
Shifting the graph of a function

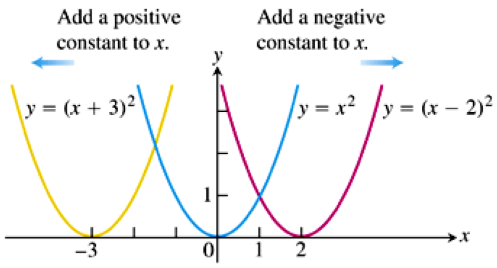
Suppose f is a function and $c \in \mathbb{R}$. Let g and h be two new functions defined by $g(x) = f(x) + c$ and $h(x) = f(x + c)$. Then

- the graph of g is equal to the graph of f shifted up by c units.
- the graph of h is equal to the graph of f shifted to the left by c units.

Note that if $c < 0$ then a shift up by c units is actually a shift down, and a shift to the left by c units is actually a shift to the right. Note also that g and h can both be obtained from f by taking a composition with a linear function: if $k(x) = x + c$ for all $x \in \mathbb{R}$ then $g = k \circ f$ and $h = f \circ k$.

Example:



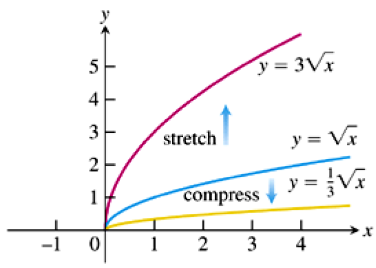


Scaling and reflecting the graph of a function

Suppose f is a function and $c\mathbb{R}$. Let g and h be two new functions defined by $g(x) = cf(x)$ and $h(x) = f(cx)$. If $c > 0$ then

- the graph of g is equal to the graph of f scaled by a factor of c along the y-axis.
- the graph of h is equal to the graph of f scaled by a factor of c along the x-axis.

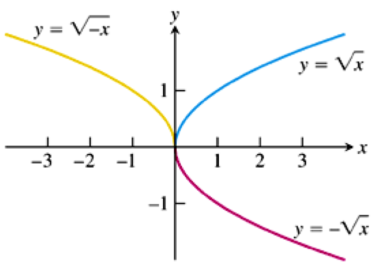
Example



If $c = -1$ then

- the graph of g is equal to the graph of f reflected across the x-axis.
- the graph of h is equal to the graph of f reflected across the y-axis.

Example



More generally, when $c < 0$, we get a combination of a scaling and a reflection - see Exercise Sheet 2.