MTH4100 Calculus I

Lecture notes for Week 1

Thomas' Calculus, Appendix 1

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What is Calculus?

Calculus is the branch of mathematics which uses *limits, derivatives and integrals* to 'measure change'. It is based on the *real numbers* and the study of *functions* of real variables:

- for one variable see Calculus I
- for several variables see Calculus II

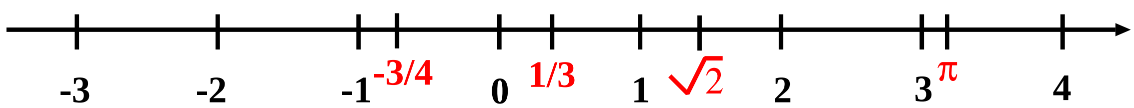
Calculus provides powerful techniques for solving problems which have widespread applications throughout science, economics, and engineering. It has been formalised and extended into the important branch of mathematics known as *analysis*.

Real numbers and the real line

We can think of the real numbers as the *set* of all infinite decimals. We denote this set by $\mathbb{R}$.[1]

examples: $2 = 2.000\ldots$; $-\frac{3}{4} = -0.7500\ldots$; $\frac{1}{3} = 0.333\ldots$; $\sqrt{2} = 1.4142\ldots$; $\pi = 3.1415\ldots$

The real numbers can be represented as points on the *real line*:



The real numbers have three types of fundamental *properties*:

- **algebraic**: the rules of calculation (addition, subtraction, multiplication, division). Example: $2(3+5) = 2\cdot 3 + 2 \cdot 5 = 6 + 10 = 16$
- **order**: inequalities relating any two real numbers (for a geometric picture imagine the order in which points occur on the real line). Example: $-\frac{3}{4} < \frac{1}{3}$, $\quad \sqrt{2} \leq \pi$
- **completeness**: "there are no gaps on the real line"

1. Algebraic properties.
The first five algebraic properties involve *addition*:
(A0) For all $a, b \in \mathbb{R}$ we have $a + b \in \mathbb{R}$. *closure*
(A1) For all $a, b, c \in \mathbb{R}$ we have $a + (b + c) = (a + b) + c$. *associativity*
(A2) For all $a, b \in \mathbb{R}$ we have $a + b = b + a$. *commutativity*
(A3) There is an element $0 \in \mathbb{R}$ such that $a + 0 = a$ for all $a \in \mathbb{R}$. *identity*
(A4) For all $a \in \mathbb{R}$ there is an element $-a \in \mathbb{R}$ such that $a + (-a) = 0$. *inverse*

Why these rules? They define an algebraic structure called an *abelian*, (or *commutative*), *group*. Other examples of abelian groups are the *integers* $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and the *rational numbers* $\mathbb{Q} = \{m/n \,:\, m, n \in \mathbb{Z} \text{ and } n \neq 0\}$.[2]

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[1] A *set* is just a 'collection of objects'. The objects in a set are called *elements* and we write $x \in \mathbb{R}$ as shorthand for 'x is an element of $\mathbb{R}$' i.e. x is a real number. You will learn more about sets in MTH4110 Mathematical Structures.

[2] This set theory notation is shorthand for saying "$\mathbb{Q}$ is the set of all numbers m/n such that m, n are integers and $n \neq 0$."

We have five analogous algebraic properties for *multiplication*:
(M0) For all $a, b \in \mathbb{R}$ we have $ab \in \mathbb{R}$ *closure*
(M1) For all $a, b, c \in \mathbb{R}$ we have $a(bc) = (ab)c$ *associativity*
(M2) For all $a, b \in \mathbb{R}$ we have $ab = ba$ *commutativity*
(M3) There is an element $1 \in \mathbb{R}$ such that $a\,1 = a$ for all $a \in \mathbb{R}$. *identity*
(M4) For all $a \in \mathbb{R}$ with $a \neq 0$, there is an element $a^{-1} \in \mathbb{R}$ such that $a\,a^{-1} = 1$ *inverse.*

Note that the rationals $\mathbb{Q}$ also satisfy properties M0-M4 but the integers $\mathbb{Z}$ do not.

One final algebraic property connects *multiplication with addition*:

(D) For all $a, b, c \in \mathbb{R}$ we have $a(b + c) = ab + ac$ *distributivity*

These 11 rules define an algebraic structure called a *field*. Since the reals $\mathbb{R}$ and rationals $\mathbb{Q}$ both satisfy all 11 properties, each of them is an example of a field.

2. Order properties
For all $a, b, c \in \mathbb{R}$ we have:
(O1) either $a \leq b$ or $b \leq a$ *totality of ordering I*
(O2) if $a \leq b$ and $b \leq a$ then $a = b$ *totality of ordering II*
(O3) if $a \leq b$ and $b \leq c$ then $a \leq c$ *transitivity*
(O4) if $a \leq b$ then $a + c \leq b + c$ *order under addition*
(O5) if $a \leq b$ and $0 \leq c$ then $a\,c \leq b\,c$ *order under multiplication*

Properties A0-A4, M0-M4, D, and O1-O5 define a mathematical structure called an *ordered field.*

Some useful rules for calculations with inequalities (practise in exercises) are:

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \dfrac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \dfrac{1}{b} < \dfrac{1}{a}$

We can *prove* that these rules are valid by using using properties (O1) to (O5): 1. to 3. are straightforward, 4. to 6. are more tricky.

3. Completeness property This is more difficult to explain. Intuitively it means "there are no gaps in the real numbers". More precisely it says: if a set of real numbers S has an *upper bound* i.e. there exists a number $c \in \mathbb{R}$ such that $x \leq c$ for all $x \in S$, then S has a *least upper bound* i.e. there exists an upper bound c_0 for S such that $c \geq c_0$ for all upper bounds c of S. This property may seem obvious, *but it does not hold for the rational numbers*. Consider for example the set $S = \{q \in \mathbb{Q} : q^2 \leq 2\}$. This set has an upper bound in $\mathbb{Q}$, for example $c = 3/2$. But it has no *least* upper bound in $\mathbb{Q}$. The problem is that the only possible contender for a least upper bound for S is $c_0 = \sqrt{2}$ and $\sqrt{2} \notin \mathbb{Q}$. We will "prove" this last statement:

Theorem 1 *$x^2 = 2$ has no solution for $x \in \mathbb{Q}$.*

Proof: We use *proof by contradiction.* Assume there is an $x \in \mathbb{Q}$ with $x^2 = 2$. Then x must be of the form $x = \frac{m}{n}$, $m, n \in \mathbb{Z}$, $n \neq 0$. We can assume that **m and n have no common factors** (otherwise we can cancel them).
Now $x^2 = 2$ implies that $(\frac{m}{n})^2 = 2$, so $m^2 = 2n^2$, and hence m^2 is even.
However, the fact that m^2 even implies that **m is even**. Writing $m = 2m_1$ we have $2n^2 = m^2 = (2m_1)^2 = 4m_1^2$, and hence $2m_1^2 = n^2$.
This implies that n^2 is even, so **n is even** as well.
We have now shown that both m and n must be even, and hence they **share a common factor 2**.
This is a contradiction! Therefore the assumption that there is an $x \in \mathbb{Q}$ with $x^2 = 2$ must be false. •

We do not have the mathematical tools to discuss completeness any further. It is covered in MTH5104 Convergence and Continuity, a 2nd year "analysis" module.

University mathematics is built upon

- basic properties (Definitions, Axioms)
- statements (Lemmas, Propositions, Theorems, Corollaries, ...) which we deduce from the basic properties by giving *proofs*!

You will learn how to write your own proofs in MTH4110 Mathematical Structures. The calculus modules MTH4100 and MTH4101 will be primarily concerned with using calculus to solve problems.

Intervals

Definition An *interval* is a subset I of $\mathbb{R}$ of one of the following two types:

(a) all real numbers which lie between two given real numbers;

(b) all real numbers which are either above or below a given real number.

Type (a) intervals are said to be *bounded* (or *finite*). Type (b) intervals are said to be *unbounded* (or *infinite*). The completeness property tells us that an interval which is bounded above has a least upper bound. Similarly an interval which is bounded below has a greatest lower bound. We refer to these values as *end-points* of the interval.
Examples:

- $I = \{x \in \mathbb{R} : 3 < x \leq 6\}$ defines a bounded interval. Geometrically, it corresponds to a *line segment* on the real line. It has two end-points 3 and 6. We can describe it using the notation $I = (3, 6]$, where the round bracket on the left tells us that $3 \notin I$ and the square bracket on the right tells us that $6 \in I$.

- $I = \{x \in \mathbb{R} : x > -2\}$ defines an unbounded interval. Geometrically, it corresponds to a *ray* i.e. a line which extends to infinity in one direction. It has one end-point -2. We can describe it using the notation $I = (-2, \infty)$.

We can distinguish between intervals which are bounded or unbounded. We can also distinguish between intervals by considering whether or not they contain their end points: intervals which contain all their end-points are *closed*; intervals which contain none of their end-points are *open*; intervals which have two end points and contain exactly one of them are *half-open* (or *half-closed*).

TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \leq x \leq b\}$	Closed	
	$[a, b)$	$\{x \mid a \leq x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \leq b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \geq a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \leq b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

Solving inequalities We can represent the set of all solutions to one or more inequalities as an interval or, more generally, as a collection of disjoint intervals.

Examples: Find the set of all solutions to the following inequalities.

(a) $2x - 1 < x + 3$. Using the properties of order we have $2x < x + 4$ and hence $x < 4$. Thus the set of solutions is the interval $(-\infty, 4)$.

(b) $\frac{6}{x-1} \geq 5$. Since $\frac{6}{x-1} > 0$ we have $x - 1 > 0$ and hence $x > 1$. We can now use property (O5) to deduce that $6 \geq 5x - 5$ and hence $\frac{11}{5} \geq x$. Combining these two inequalities we see that the set of solutions is the interval $(1, \frac{11}{5}]$.

(c) $x^2 - 2x - 1 > 2$. Then $x^2 - 2x - 3 > 0$ so $(x + 1)(x - 3) > 0$. Hence either $(x + 1)$ and $(x - 3)$ are both positive i.e. $x > 3$, or $(x + 1)$ and $(x - 3)$ are both negative i.e. $x < -1$. Thus the set of solutions is union of the two disjoint intervals $(-\infty, -1)$ and $(3, \infty)$.

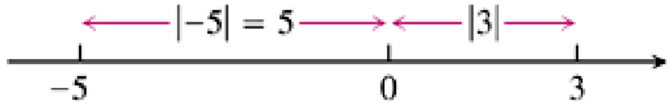
Absolute value

Definition The *absolute value* (or *modulus*) of a real number x is defined as:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

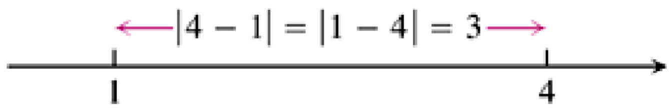
Geometrically, $|x|$ is the distance on the real line between x and 0.

Example:



Similarly, for any $x, y \in \mathbb{R}$, $|x-y|$ is the distance between x and y.

Example:



Lemma 1 (Properties of Absolute value) *Suppose* $a, b \in \mathbb{R}$*. Then:*

1. $|a| = \sqrt{a^2}$;
2. $|-a| = |a|$;
3. $|ab| = |a|\,|b|$;
4. $|\frac{a}{b}| = \frac{|a|}{|b|}$ *when* $b \neq 0$;
5. $|a+b| \leq |a| + |b|$, *the* triangle inequality.

We need to prove these statements.
Proof of (1). By definition, the symbol $\sqrt{a^2}$ is always taken to be the non-negative square root of a^2. So $\sqrt{a^2} = a$ if $a \geq 0$ and $\sqrt{a^2} = -a$ if $a < 0$. Hence $|a| = \sqrt{a^2}$.
We will use (1) to prove (2)-(5).
Proof of (2). We have

$$|-a| = \sqrt{(-a)^2} = \sqrt{a^2} = |a|.$$

We have used a *direct proof*: We started on the left hand side of the equation and transformed it step by step until we arrived at the right hand side.
Proof of (3) We have

$$|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a|\,|b|.$$

Proof of (4): exercise!

Proof of (5) We use a trick. We first show that $|a+b|^2 \le (|a|+|b|)^2$. Since $|a+b| = \sqrt{(a+b)^2}$ we have

$$\begin{aligned} |a+b|^2 &= \left(\sqrt{(a+b)^2}\right)^2 \\ &= (a+b)^2 \\ &= a^2+2ab+b^2 \\ &\le a^2+2|a|\,|b|+b^2 \qquad \text{(because } ab \le |ab| = |a||b| \text{ by (2))} \\ &= |a|^2+2|a|\,|b|+|b|^2 \\ &= (|a|+|b|)^2 \end{aligned}$$

Taking the square roots of both sides and using the facts that $|a+b|$ and $|a|+|b|$ are both non-negative we may deduce that (5) holds. •

Absolute Values and Intervals We can represent the set of all solutions to inequalities involving absolute values as unions of one or more disjoint intervals.

Lemma 2 (Absolute values and Intervals) *Suppose a is a positive real number. Then:*

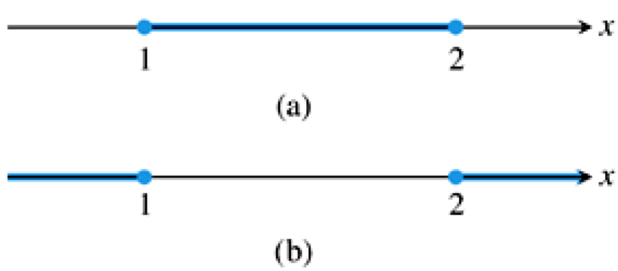
1. $|x| = a \Leftrightarrow x = \pm a$;[3]
2. $|x| < a \Leftrightarrow -a < x < a \Leftrightarrow x \in (-a, a)$;
3. $|x| > a \Leftrightarrow x < -a$ *or* $x > a \Leftrightarrow x \in (-\infty, -a) \cup (a, \infty)$;
4. $|x| \le a \Leftrightarrow -a \le x \le a \Leftrightarrow x \in [-a, a]$;
5. $|x| \ge a \Leftrightarrow x \le -a$ *or* $x \ge a \Leftrightarrow x \in (-\infty, -a] \cup [a, \infty)$.

Proof of (4). This follows because the distance from x to 0 is less than or equal to a if and only if x lies between a and $-a$. •

Examples

(a) $|2x-3| \le 1$ if and only if $-1 \le 2x-3 \le 1$ i.e. $x \in [1, 2]$.

(b) $|2x-3| \ge 1$ if and only if $2x-3 \le -1$ or $2x-3 \ge 1$ i.e. $x \in (-\infty, 1]$ or $x \in [2, \infty)$.



Reading Assignment: read

Thomas' Calculus, Appendix 3: Lines, Circles, and Parabolas

[3]The symbol $\Leftrightarrow$ is shorthand for 'if and only if'. It is used to link equivalent statements.