MTH4100 Calculus I

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Week 12, Semester 1, 2012

Irrational powers of real numbers

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Suppose a is a positive real number and $q \in \mathbb{Q}$. Then

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This definition implies that

$$\ln(a^x) = \ln[\exp(x \ln a)] = x \ln a$$

for all $x \in \mathbb{R}$.

Properties of the exponential function with base a

Lemma

Suppose $a, b, c \in \mathbb{R}$ *and* $a > 0$. *Then:*

1. $a^b \cdot a^c = a^{b+c}$;
2. $(a^b)^c = a^{bc}$.

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Theorem

Suppose that $a > 0$ and $a \neq 1$. Then the exponential function with base a is differentiable for all $x \in \mathbb{R}$ and satisfies

$$\frac{d}{dx} a^x = a^x \ln a \,.$$

Hence

$$\int a^x \, dx = \frac{a^x}{\ln a} + C \,.$$

Logarithms to the base a

Definition When $a > 1$, $\frac{d}{dx}a^x = a^x \ln a > 0$ and hence $f(x) = a^x$ is strictly increasing for all $x \in \mathbb{R}$.

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This implies that $f(x) = a^x$ is injective for any fixed $a > 0$ with $a \neq 1$. Hence its inverse function exists. This inverse function is called the *logarithm of x to the base a* and is denoted by $\log_a x$.

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We have

$$\log_a(a^x) = x = a^{\log_a x}$$

for all $x \in \mathbb{R}$, by the definition of an inverse function.

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This gives

$$\ln x = \ln\left(a^{\log_a x}\right) = \log_a x \cdot \ln a\,.$$

and hence

$$\log_a x = \frac{\ln x}{\ln a}$$

Further properties of the exponential function

The definition of a^x gives us an *alternative notation* for $\exp(x)$. Recall that $1 = \ln e$ where e is Euler's constant. This implies that

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We have seen that $\frac{d}{dx}e^x = e^x$. This gives $\int e^x dx = e^x + C$. We can now use the chain rule to deduce:

Lemma

Let $f(x)$ be a differentiable function. Then

$$\frac{d}{dx}e^{f(x)} = e^{f(x)} f'(x)$$

and

$$\int e^{f(x)} f'(x) dx = e^{f(x)} + C.$$

The number e as a limit

Theorem

$$e = \lim_{x \to 0} (1 + x)^{1/x}$$

Techniques of Integration

- Basic formulas, see integration tables (Thomas' Calculus, page 435 and more extensive tables pages T1-T6)
- Procedures for matching integrals to basic formulas
- Other techniques (substitution, integration by parts, partial fractions)

This needs practice, practice, practice, . . .:

Exercise sheet 10 and online exercises sets 9,10

Basic formulas

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C$ (any number k)
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C$ $(n \neq -1)$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C$
    $= -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \frac{a^u}{\ln a} + C$ $(a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$
21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C$ $(a > 0)$
22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C$ $(u > a > 0)$

Procedures for matching integrals to basic formulas

Procedures for Matching Integrals to Basic Formulas

PROCEDURE	EXAMPLE
Making a simplifying substitution	$\dfrac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \dfrac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$
Using a trigonometric identity	$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x\tan x + \tan^2 x$ $= \sec^2 x + 2\sec x \tan x + (\sec^2 x - 1)$ $= 2\sec^2 x + 2\sec x\tan x - 1$
Eliminating a square root	$\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$
Reducing an improper fraction	$\dfrac{3x^2-7x}{3x+2} = x - 3 + \dfrac{6}{3x+2}$
Separating a fraction	$\dfrac{3x+2}{\sqrt{1-x^2}} = \dfrac{3x}{\sqrt{1-x^2}} + \dfrac{2}{\sqrt{1-x^2}}$
Multiplying by a form of 1	$\sec x = \sec x \cdot \dfrac{\sec x + \tan x}{\sec x + \tan x}$ $= \dfrac{\sec^2 x + \sec x \tan x}{\ldots}$

Integration by parts

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$$\int u\,dv = uv - \int v\,du \qquad (2)$$

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Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \qquad (3)$$

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Example: Evaluate

$$\int x \cos x \, dx \quad :$$

Integration by parts - general strategy

- Choose u such that du is "simpler" than u;
- Choose dv such that vdu is easy to integrate;
- If your result looks more complicated after doing integration by parts, it's most likely not right. Try something else.

Integration by parts - general strategy

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Read Thomas' Calculus, Section 8.1, examples 3 to 6 and practice by doing online exercise set 10

The method of partial fractions

Example 1: If we know that

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}$$

then we can easily integrate

$$\begin{aligned} \int \frac{5x-3}{x^2-2x-3}dx &= \int \frac{2}{x+1}dx + \int \frac{3}{x-3}dx \\ &= 2\ln|x+1| + 3\ln|x-3| + C \end{aligned}$$

To obtain such simplifications, we use the *method of partial fractions*.

Example 2 (repeated linear factor). Find

$$\int \frac{6x+7}{(x+2)^2}\,dx\ .$$

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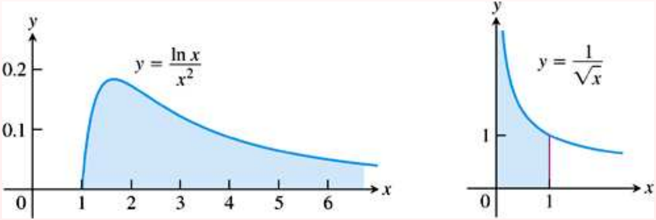
Improper integrals

Can we compute areas under *infinitely extended curves*?
Two examples of improper integrals:



Type 1: area extends from $x = 1$ to $x = \infty$.
Type 2: area extends from $x = 0$ to $x = 1$ *but* $f(x)$ only defined on $(0, 1]$.

Improper integrals - Examples

Example of type 1: Find

$$\int_0^\infty e^{-x/2}\,dx$$

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