MTH4100 Calculus I

Lecture notes for Week 12

Thomas' Calculus, Sections 8.1 to 8.3, and 8.7

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Irrational powers of real numbers

We have defined what we mean by a^q for any real number $a > 0$ and any *rational* number q. We can use the exponential function to extend this to a definition of a^x when x is *irrational* i.e. $x \in \mathbb{R} \setminus \mathbb{Q}$. We first express a^q in terms of the exponential function.

Lemma 1 *Suppose a is a positive real number and $q \in \mathbb{Q}$. Then*

$$a^q = \exp(q \ln a) \,. \tag{1}$$

Proof The fourth rule for manipulating natural logarithms tells us that

$$\ln a^q = q \ln a \,.$$

Taking the exponential of both sides of this equation (and using $\exp = \ln^{-1}$) gives

$$a^q = \exp(\ln a^q) = \exp(q \ln a) \,.$$

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Since the right hand side of (1) makes sense for all $q \in \mathbb{R}$ we can use it define a^x for all *real* numbers x.

Definition For any $a \in \mathbb{R}$ with $a > 0$, the *exponential function with base* a, a^x is defined by putting

$$a^x = \exp(x \ln a)$$

for all $x \in \mathbb{R}$.

Note that this definition implies that

$$\ln(a^x) = \ln[\exp(x \ln a)] = x \ln a \tag{2}$$

and hence that the fourth rule for manipulating natural logarithms holds for all powers of a, not just rational powers.

For the definition of a^x to make sense we will need the exponent in a^x to behave in the same way as exponents for integer or rational powers of a. This follows from our next result.

Lemma 2 *Suppose a is a positive real number and $b, c \in \mathbb{R}$. Then:*

1. $a^b \cdot a^c = a^{b+c}$ *:*
2. $(a^b)^c = a^{bc}$ *.*

Proof By definition $a^b = \exp(b \ln a)$ and $a^c = \exp(c \ln a)$. Hence

$$\begin{aligned} a^b \cdot a^c &= \exp[\ln(a^b \cdot a^c)] \\ &= \exp[\ln(a^b) + \ln(a^c)] \qquad \text{(by the first rule for manipulating logs)} \\ &= \exp[b \ln(a) + c \ln(a)] \qquad \text{(by 2)} \\ &= \exp[(b+c)\ln(a)] \\ &= a^{b+c}. \end{aligned}$$

Similarly

$$\begin{aligned} (a^b)^c &= \exp(c \ln a^b) \\ &= \exp(cb \ln(a)) \\ &= a^{bc}. \end{aligned}$$

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Note: The exponential function with base a is differentiable for all $x \in \mathbb{R}$ and

$$\frac{d}{dx} a^x = \frac{d}{dx} \exp(x \ln a) = \exp(x \ln a) \cdot \ln a = a^x \ln a$$

by the chain rule. Hence

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

when $a > 0$ and $a \neq 1$.

Definition When $a > 1$, $\frac{d}{dx} a^x = a^x \ln a > 0$ and hence $f(x) = a^x$ is strictly increasing for all $x \in \mathbb{R}$. When $0 < a < 1$, a similar argument shows that $f(x) = a^x$ is strictly decreasing for all $x \in \mathbb{R}$. This implies that $f(x) = a^x$ is injective for any fixed $a > 0$ with $a \neq 1$. Hence its inverse function exists. This inverse function is called the *logarithm of x to the base a* and is denoted by $\log_a x$. We have

$$\log_a(a^x) = x = a^{\log_a x}$$

for all $x \in \mathbb{R}$. This gives

$$\ln x = \ln\left(a^{\log_a x}\right) = \log_a x \cdot \ln a\,.$$

and hence

$$\log_a x = \frac{\ln x}{\ln a}$$

Note: The algebra for $\log_a x$ is precisely the same as that for $\ln x$.

Further properties of the exponential function

The above definition of a^x gives us an *alternative notation* for $\exp(x)$. Recall that $1 = \ln e$ where e is Euler's constant. This implies that

$$e^x = \exp(x \ln e) = \exp x.$$

Henceforth we will often use e^x instead of $\exp x$.

We have seen that $\frac{d}{dx}e^x = e^x$. This gives

$$\int e^x dx = e^x + C\,.$$

We can now use the chain rule to deduce:

Lemma 3 *Let $f(x)$ be a differentiable function. Then*

$$\frac{d}{dx}e^{f(x)} = e^{f(x)} f'(x)$$

and

$$\int e^{f(x)} f'(x) dx = e^{f(x)} + C\,.$$

Examples:

1.
$$\frac{d}{dx}e^{\sin x} = e^{\sin x}\frac{d}{dx}\sin x = e^{\sin x}\cos x$$

2.
$$\begin{aligned}\int_0^{\ln 2} e^{3x} dx &= \int_0^{\ln 8} e^u \frac{1}{3} du \\ &= \left.\frac{1}{3}e^u\right|_0^{\ln 8} \\ &= \frac{7}{3}\end{aligned}$$

We defined e via $\ln e = 1$ and stated $e = 2.718281828459\ldots$.

Theorem 1 (The number e as a limit)

$$e = \lim_{x\to 0}(1+x)^{1/x}$$

Proof We have

$$\begin{aligned}\ln\left(\lim_{x\to 0}(1+x)^{1/x}\right) &= \lim_{x\to 0}\left(\ln(1+x)^{1/x}\right) && \text{(continuity of } \ln x \text{)} \\ &= \lim_{x\to 0}\left(\frac{1}{x}\ln(1+x)\right) && \text{(power rule)} \\ &= \lim_{x\to 0}\frac{1}{1+x} && \text{(l'Hôpital)} \\ &= 1\end{aligned}$$

Taking exponentials of both sides gives:

$$\lim_{x\to 0}(1+x)^{1/x} = \exp\left[\ln\left(\lim_{x\to 0}(1+x)^{1/x}\right)\right] = \exp 1 = e\,.$$

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Techniques of Integration

- Basic properties (Thomas' Calculus, Chapter 5)
- Basic formulas, see integration tables (Thomas' Calculus, page 435 and more extensive tables on pages T1-T6)
- Procedures for matching integrals to basic formulas
- Other techniques (substitution, integration by parts, partial fractions)

This needs practice, practice, practice, ...:

Exercise sheet 10 and online exercise sets 9 and 10

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C \qquad$ (any number k)
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \dfrac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{u}{a}\right| + C$
21. $\int \dfrac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (a > 0)$
22. $\int \dfrac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (u > a > 0)$

Procedures for Matching Integrals to Basic Formulas

PROCEDURE	EXAMPLE
Making a simplifying substitution	$\dfrac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \dfrac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$
Using a trigonometric identity	$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x\tan x + \tan^2 x$ $= \sec^2 x + 2\sec x \tan x + (\sec^2 x - 1)$ $= 2\sec^2 x + 2\sec x\tan x - 1$
Eliminating a square root	$\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$
Reducing an improper fraction	$\dfrac{3x^2-7x}{3x+2} = x - 3 + \dfrac{6}{3x+2}$
Separating a fraction	$\dfrac{3x+2}{\sqrt{1-x^2}} = \dfrac{3x}{\sqrt{1-x^2}} + \dfrac{2}{\sqrt{1-x^2}}$
Multiplying by a form of 1	$\sec x = \sec x \cdot \dfrac{\sec x + \tan x}{\sec x + \tan x}$ $= \dfrac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}$

See exercise sheet 10 and online exercise set 9 for further examples

Integration by parts

We have seen that the chain rule for differentiation gives rise to the substitution law for integration. The technique of integration by parts can be derived from the product rule for differentiation in a similar way. For any two differentiable functions f and g we have

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)\,.$$

Integrate both sides of this equation gives

$$\int \frac{d}{dx}\left(f(x)g(x)\right)dx = \int \left(f'(x)g(x) + f(x)g'(x)\right)dx\,.$$

Therefore,

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

leading to

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Putting $u = f(x)$ and $v = g(x)$ this formula can be abbreviated to

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

Similarly

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \qquad (3)$$

Example: Evaluate

$$\int x\cos x\,dx\ :$$

Let $u = x$ and $dv = \cos x\,dx$. Then $du = dx$ and $v = \sin x$. The integration by parts formula now gives:

$$\begin{aligned}\int x\cos x\,dx &= x\sin x - \int \sin x\,dx\\ &= x\sin x + \cos x + C \qquad \text{(Do not forget the constant } C\text{).}\end{aligned}$$

Let's explore the four possible choices of u and dv for $\int x\cos x\,dx$:

1. $u = 1$, $dv = x\cos x\,dx$:
   We don't know of how to compute$\int dv$: no good!

2. $u = x$ and $dv = \cos x\,dx$:
   Done above, works!

3. $u = \cos x$, $dv = x\,dx$:
   Now $du = -\sin x\,dx$ and $v = x^2/2$ so that

   $$\int x\cos x\,dx = \frac{1}{2}x^2\cos x + \int \frac{1}{2}x^2\sin x\,dx$$

   This makes the situation worse!

4. $u = x\cos x$ and $dv = dx$:
   Now $du = (\cos x - x\sin x)dx$ and $v = x$ so that

$$\int x\cos x\, dx = x^2\cos x - \int x(\cos x - x\sin x)dx$$

   This again is worse!

General advice:

- Choose u such that du is "simpler" than u;
- Choose dv such that vdu is easy to integrate;
- If your result looks more complicated after doing integration by parts, it's most likely not right. Try something else.

Read Thomas' Calculus:
Section 8.1, examples 3 to 6:
Four further examples of integration by parts...
...and practice by doing online exercise set 10

The method of partial fractions

Example: If we know that

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}$$

then we can easily integrate

$$\begin{aligned}\int \frac{5x-3}{x^2-2x-3}dx &= \int \frac{2}{x+1}dx + \int \frac{3}{x-3}dx \\ &= 2\ln|x+1| + 3\ln|x-3| + C\end{aligned}$$

To obtain such simplifications, we use the *method of partial fractions.*
Let $f(x)/g(x)$ be a rational function, for example,

$$\frac{f(x)}{g(x)} = \frac{2x^3-4x^2-x-3}{x^2-2x-3}$$

If $\deg(f) \geq \deg(g)$, we first use polynomial division:

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3}$$

and consider the remainder term. We also have to know the factors of $g(x)$:

$$x^2-2x-3 = (x+1)(x-3)$$

Now we can write

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

and obtain

$$5x-3 = A(x-3) + B(x+1) = (A+B)x + (-3A+B).$$

We can now equate the coefficients of the same powers of x to obtain $A+B=5$ and $-3A+B=3$. Solving these two simultaneous equations gives $A=2$, and $B=3$ as above.[1]

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x-r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x-r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let x^2+px+q be a quadratic factor of $g(x)$. Suppose that $(x^2+px+q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx+C_n}{(x^2+px+q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example (of a repeated linear factor). Find

$$\int \frac{6x+7}{(x+2)^2} dx \; .$$

- Write

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} \; .$$

- Multiply by $(x+2)^2$ to get

$$6x+7 = A(x+2) + B = Ax + (2A+B) \; .$$

[1] For *this* example we could also substitute $x=-1$ in the equation $5x-3 = A(x-3)+B(x+1)$ to obtain $A=2$, and substitute $x=3$ to obtain $B=3$.

- Equate coefficients of the same powers of x and solve:

$$A = 6 \text{ and } 2A + B = 7 \Rightarrow B = -5 \; .$$

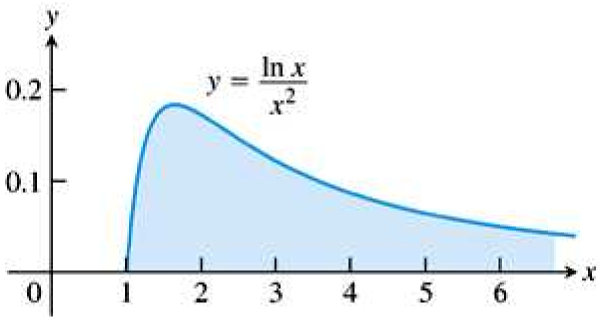
- Integrate:

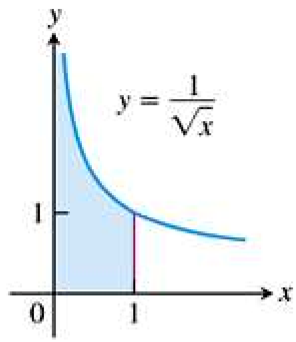
$$\int \frac{6x+7}{(x+2)^2} dx = 6 \int \frac{dx}{x+2} - 5 \int \frac{dx}{(x+2)^2} = 6 \ln|x+2| + 5(x+2)^{-1} + C \; .$$

Read Thomas' Calculus:
Section 8.4, examples 1, 4 and 5:
Three more advanced examples...
...and practice by doing online exercise set 10.

Improper integrals

Can we compute areas under *infinitely extended curves*?
Two examples of improper integrals:

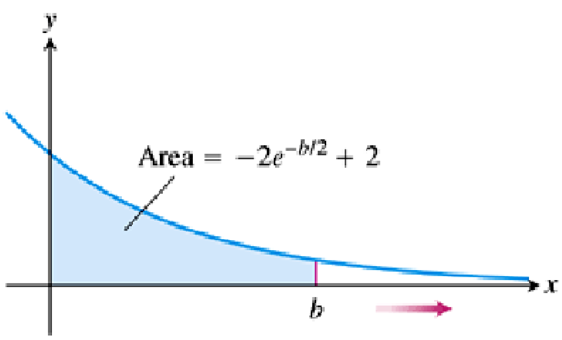




Type 1: area extends from $x = 1$ to $x = \infty$.
Type 2: area extends from $x = 0$ to $x = 1$ *but* $f(x)$ diverges at $x = 0$.

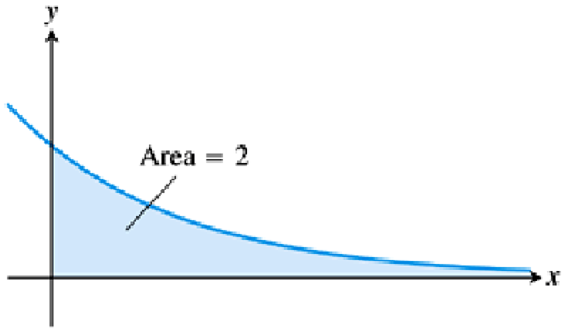
Calculation of type I improper integrals in two steps.

Example: $y = e^{-x/2}$ on $[0, \infty)$
1. Calculate bounded area:



$$A(b) = \int_0^b e^{-x/2} dx = -2e^{-x/2}\Big|_0^b = -2e^{-b/2} + 2$$

2. Take the limit:



$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty}(-2e^{-b/2} + 2) = 2$$

$$\Rightarrow \int_0^\infty e^{-x/2} dx = 2$$

DEFINITION **Type I Improper Integrals**

Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

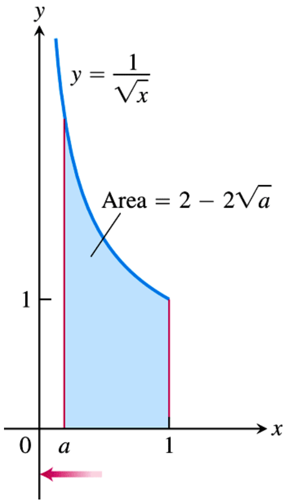
3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Calculation of type II improper integrals in two steps.
Example: $y = 1/\sqrt{x}$ on $(0, 1]$



1. Calculate bounded area:

$$A(a) = \int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_a^1 = 2 - 2\sqrt{a}$$

2. Take the limit:

$$\lim_{a\to 0^+} A(a) = \lim_{a\to 0^+} (2 - 2\sqrt{a}) = 2$$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{x}} = 2$$

DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

Read Thomas' Calculus:
Section 8.7, examples 1 to 5:
Five more examples...
...and practice by doing online exercise set 10.