MTH4101 Calculus II

Lecture notes for Week 12
A First Look at Differential Equations

Thomas' Calculus, Sections 7.4, 9.1 and 9.2

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Example:
Solve the initial value problem

$$\frac{dy}{dx} = (1+y)e^x \ , \ y > -1 \ , \ y(0) = 0 \ .$$

By separation of variables and integration we obtain

$$\begin{aligned} \int \frac{dy}{1+y} &= \int e^x dx \\ \ln(1+y) &= e^x + C \ , \ C = \text{const.} \ , \end{aligned}$$

which gives the solution y in *implicit form*. The constant C is determined by using the initial condition $y(0) = 0$:

$$\ln(1+0) = 0 = e^0 + C = 1 + C$$

giving $C = -1$. The *explicit solution* of the initial value problem is obtained as

$$y(x) = e^{e^x - 1} - 1 \ .$$

First-order linear differential equations and the integrating factor

A first-order **linear** differential equation is one that can be written in the **standard form**

$$\frac{dy}{dx} + P(x)y = Q(x) \ , \qquad (1)$$

where $P = P(x)$ and $Q = Q(x)$ are continuous functions of x. It is linear (in y), because y and its derivative dy/dx occur only to the first power, they are not multiplied together, nor do they appear as the argument of a function (such as $\sin y, \exp(y)$, etc.).

Example:
Put the equation

$$x\frac{dy}{dx} = x^2 + 3y \ , \ x > 0 \ ,$$

in standard form.

$$\begin{aligned} \frac{dy}{dx} &= x + \frac{3}{x}y \\ \frac{dy}{dx} - \frac{3}{x}y &= x \ . \end{aligned}$$

Hence, $P(x) = -3/x$ and $Q(x) = x$.

An equation in standard form can be solved as follows: Multiply it by a function $v = v(x)$,

$$v\frac{dy}{dx} + vPy = vQ \ .$$

Now let's play a little trick: *If we choose v such that it transforms the left-hand side into the derivative of the product vy*, that is,

$$v\frac{dy}{dx} + vPy = \frac{d}{dx}(vy) \ ,$$

we can write

$$\frac{d}{dx}(vy) = vQ$$

and easily solve by integration:

$$\begin{aligned} vy &= \int vQ\,dx \\ y &= \frac{1}{v}\int vQ\,dx\,. \end{aligned} \tag{2}$$

We call $v(x)$ an **integrating factor**, because it makes the linear differential equation integrable.
Now, did we mysteriously get rid of $P(x)$ by solving our differential equation? Not quite, because we still have to determine $v(x)$ by solving the previously imposed equation

$$\frac{d}{dx}(vy) = v\frac{dy}{dx} + vPy\,.$$

Apply the product rule and simplify:

$$\begin{aligned} \frac{dv}{dx}y + v\frac{dy}{dx} &= v\frac{dy}{dx} + Pvy \\ \frac{dv}{dx}y &= Pvy\,. \end{aligned}$$

But this equation will hold if

$$\frac{dv}{dx} = Pv\,,$$

which is separable, $P = P(x)$:

$$\int \frac{dv}{v} = \int P dx$$

Without loss of generality we may assume that $v > 0$,

$$\begin{aligned} \ln v &= \int P dx \\ v &= e^{\int P dx}\,. \end{aligned} \tag{3}$$

The general solution to Eq.(1) is thus given by Eq.(2) together with Eq.(3). Note that any antiderivative of P works for Eq.(3).

Example:
Solve

$$x\frac{dy}{dx} = x^2 + 3y\,,\ x > 0\,.$$

The **integrating factor method** consists of three steps:
1. Put the equation in *standard form*:

$$\frac{dy}{dx} - \frac{3}{x}y = x\,,$$

hence $P(x) = -3/x$.

2. Calculate the *integrating factor*:

$$v = e^{\int P(x)dx} = e^{\int(-3/x)dx}$$

by choosing the simplest constant of integration, $C = 0$, and noting that $x > 0$:

$$v = e^{-3lnx} = e^{lnx^{-3}} = x^{-3}\,.$$

3. *Multiply and integrate:* Multiply both sides of the standard form by $v(x)$,

$$\frac{1}{x^3}\left(\frac{dy}{dx} - \frac{3}{x}y\right) = \frac{1}{x^3}x = \frac{1}{x^2}\,,$$

and remember that the left hand side *always* integrates into the product vy, as we have designed it to be:

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{x^3}y\right) &= \frac{1}{x^2} \\ \frac{1}{x^3}y &= \int \frac{1}{x^2}dx \\ \frac{1}{x^3}y &= -\frac{1}{x} + C\,. \end{aligned}$$

Solving this equation for y gives the general solution

$$y(x) = -x^2 + Cx^3\,,\ x > 0\,.$$

THE END