MTH4100 Calculus I

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In general, to find the *total area* between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$, do the following:

1. Draw a graph of f.
2. Subdivide $[a, b]$ at the zeros of f.
3. Integrate over each subinterval.
4. Add the *absolute* values of these integrals.

Symmetric functions

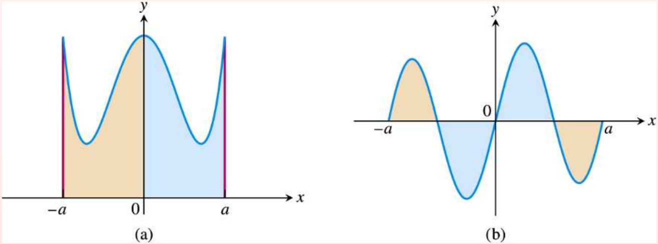
Theorem

Let f be a continuous function on the interval $[-a, a]$.

(a) *If f is even, then* $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

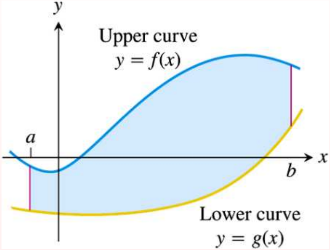
(b) *If f is odd, then* $\int_{-a}^{a} f(x)dx = 0$.

(c) *If f is either even or odd, then the* total area *between the graph of $y = f(x)$ and the x-axis over the interval $[-a, a]$ is twice the* total area *between the graph of $y = f(x)$ and the x-axis over the interval $[0, a]$.*



Areas between curves

We want to find the area between two curves $y = f(x)$ and $y = g(x)$ for $x \in [a, b]$, where $f(x) \geq g(x)$ for all $x \in [a, b]$.



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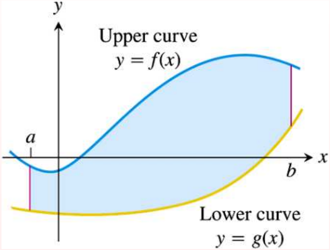

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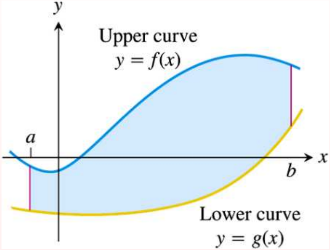

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Example: Find the area of the region R that is enclosed by the curves $y = \sqrt{x}$, $y = 0$, and $y = x - 2$.

Injective (one-to-one) functions

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Examples: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$ and $g : \mathbb{R}^+ \to \mathbb{R}$ be defined by $g(x) = \sqrt{x}$, where $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$.

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Note that if we restrict the domain of f to $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ and the domain of g to $[0, \pi/2]$ then the restricted functions will both be injective.

Definition Suppose that $f : D \to \mathbb{R}$ is an injective function with range R. Then the *inverse function* $f^{-1} : R \to D$ is defined by

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- $(f \circ f^{-1})(y) = y$ for all $y \in R$.

Method for finding inverse functions

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Step 1 Solve $y = f(x)$ for x. We have $y = x^2$ and $x \geq 0$ so $x = \sqrt{y}$. Since the domain and range of f is $\mathbb{R}^+$, we obtain

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Step 2 Relabel x and y so that y is the dependent variable and x is the independent variable. This gives:

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Relationship between graphs of f and f^{-1}

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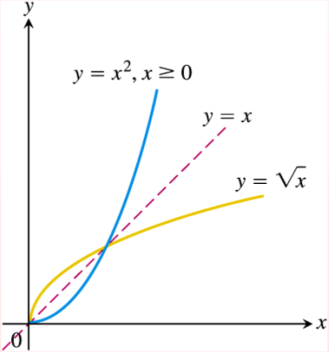
Example The graphs of $f(x) = x^2$ and $f^{-1}(x) = \sqrt{x}$.



Derivatives of inverse functions

Theorem

Suppose that $f : D \to \mathbb{R}$ *is injective, differentiable and* $f'(x) \neq 0$ *for all* $x \in D$. *Then* f^{-1} *is differentiable and its derivative* $(f^{-1})'$ *satisfies*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \, .$$

Equivalently, for all b *in the domain of* f^{-1} *we have*

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}} \, .$$

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Example $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = x^2$.

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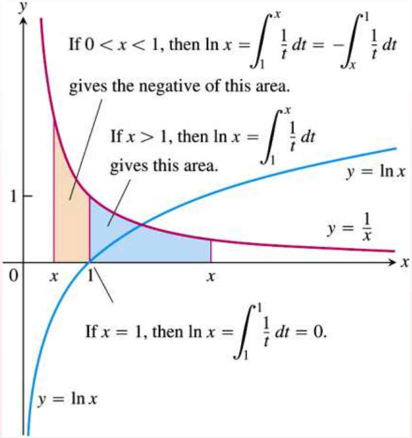
The Fundamental Theorem of Calculus (Part 1) now tells us that $F(x) = \int_1^x t^{-1} dt$ is continuous on $[a, b]$ and differentiable on (a, b) for all $0 < a < b$.
This function F is an important function: it is called the *natural logarithm function* and is denoted by ln. Thus

$$\ln x = \int_1^x t^{-1}\, dt \; .$$

Properties of the natural logarithm function



Lemma

The domain of $\ln x$ *is* $(0, \infty)$ *and its derivative is* x^{-1}.

Lemma

Suppose a, x are positive real numbers. Then

1. $\ln ax = \ln a + \ln x$.
2. $\ln \frac{1}{x} = -\ln x$.
3. $\ln \frac{a}{x} = \ln a - \ln x$.
4. $\ln x^q = q \ln x$ *for any rational number* q.

Rules for manipulating natural logarithms

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Examples:

1. $\ln 8 + \ln \cos x =$
2. $\ln \dfrac{z^2 + 3}{2z - 1} =$
3. $\ln \cot x =$
4. $\ln \sqrt[5]{x - 3} =$

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Definition The fact that the range of $\ln x$ is $(-\infty, \infty)$ implies in particular that $\ln x = 1$ for some $x \in (0, \infty)$. The point e for which $\ln e = 1$ is referred to as *Euler's constant* or *the base of the natural logarithm*. Its approximate numerical value is

$$e = 2.718281828459\ldots$$

Antiderivatives involving the natural logarithm

We have seen that $\ln x$ is an antiderivative for $1/x$ for any interval $I \subset (0, \infty)$. Our next result extends this to all intervals which do not contain zero.

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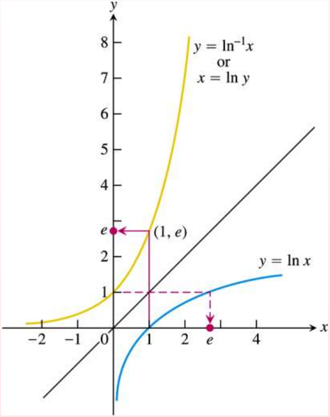
Example For $x \in (-\pi/2, \pi/2)$ we have

$$\int \tan x dx = \ln|\sec x| + C$$

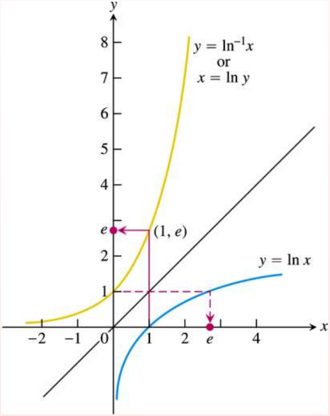
Definition The natural logarithm function is injective and hence is invertible. Its inverse function $\exp(x) = \ln^{-1}(x)$ is called the *exponential function*.



Properties of the exponential function

Lemma

The domain of $\exp x$ *is* $\mathbb{R}$ *and its range is* $(0, \infty)$. *The derivative of* $\exp x$ *is* $\exp x$.



Powers of e

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Definition For every $x \in \mathbb{R}$, put $e^x = \exp x$.

Rules for manipulating powers of e

The definition of e^x makes sense only because $e^x = \exp x$ satisfies the usual rules for powers:

Lemma

Suppose $a, b \in \mathbb{R}$. *Then*

1. $e^a \cdot e^b = e^{a+b}$
2. $e^{-a} = 1/e^a$
3. $e^a/e^b = e^{a-b}$
4. $(e^a)^b = e^{ab}$

Read

Thomas' Calculus:

Section 7.7 Inverse trigonometric functions, and Section 7.8, Hyperbolic functions

You will need this information for coursework 10!