Queen Mary
University of London

MTH4100 Calculus I

Lecture notes for Week 11

Thomas' Calculus, Sections 5.5 and 7.1 to 7.8 (except Sections 7.5, 7.6)

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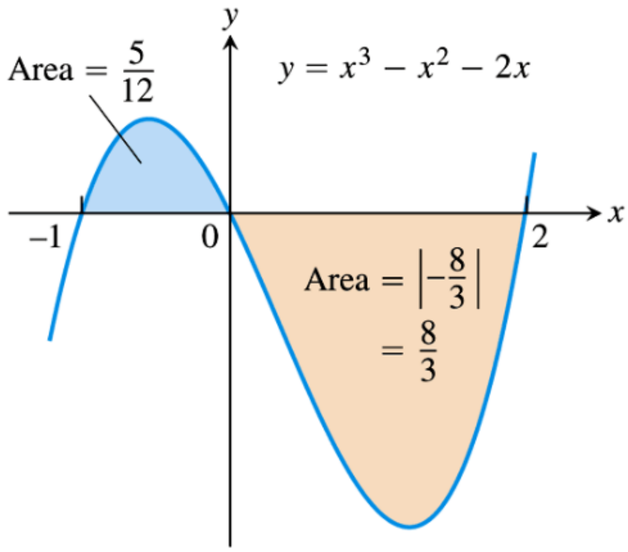
Calculating Areas

Total area

Example: Find the total area between the graph of $f(x) = x^3 - x^2 - 2x$ and the x-axis over the interval $[-1, 2]$.



1. $f(x) = x(x^2 - x - 2) = x(x+1)(x-2)$: zeros are -1, 0, 2

2.

$$\int_{-1}^{0} (x^3 - x^2 - 2x)dx = \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Big|_{-1}^{0} = \frac{5}{12}$$
$$\int_{0}^{2} (x^3 - x^2 - 2x)dx = \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Big|_{0}^{2} = -\frac{8}{3}$$

3. $A = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}$

In general, to find the *total area* between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$, do the following:

1. Draw a graph of f.
2. Subdivide $[a, b]$ at the zeros of f.
3. Integrate over each subinterval.
4. Add the *absolute* values of the integrals.

Symmetric functions

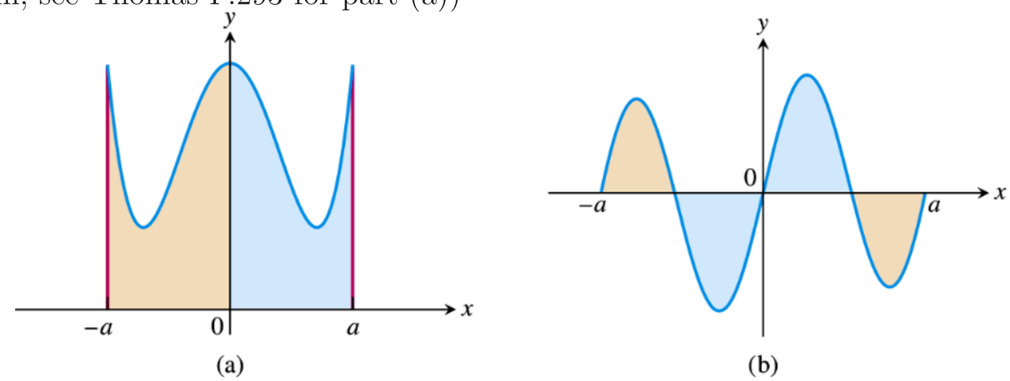
Theorem 1 *Let f be a continuous function on the interval $[-a, a]$.*

(a) *If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.*

(b) *If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.*

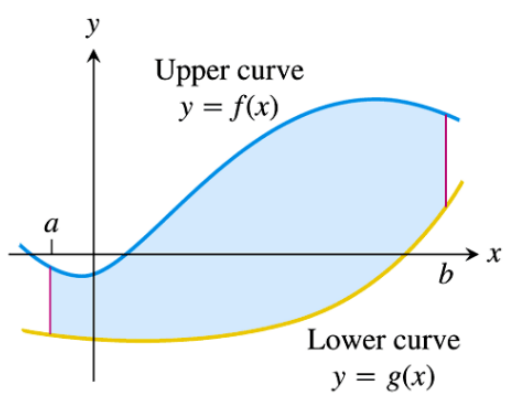
(c) *If f is either even or odd, then the* total area *between the graph of $y = f(x)$ and the x-axis over the interval $[-a, a]$ is twice the* total area *between the graph of $y = f(x)$ and the x-axis over the interval $[0, a]$.*

Proof Idea Split each integral $\int_{-a}^{a}$ into two integrals $\int_{-a}^{0} + \int_{0}^{a}$ and then manipulate the first term, see Thomas P.293 for part (a))



Areas between curves

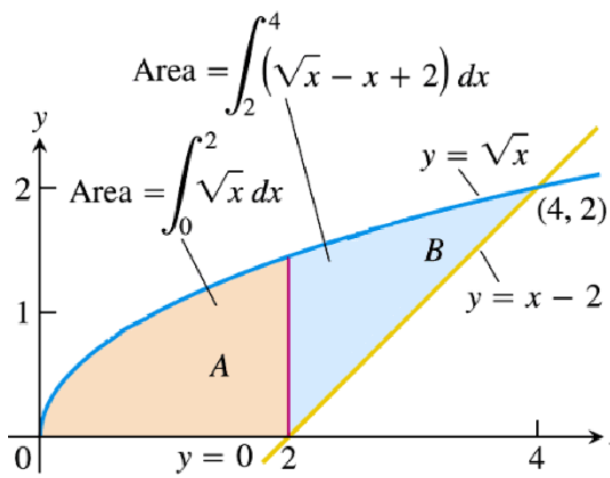
We want to find the area between two curves $y = f(x)$ and $y = g(x)$ for $x \in [a, b]$, where $f(x) \geq g(x)$ for all $x \in [a, b]$.



We can estimate this area A as a limit of Riemann sums of vertical rectangles of height $f(x) - g(x)$ and width Δx. This gives:

$$A = \int_{a}^{b} f(x) - g(x)\, dx$$

Example: Find the area of the region R that is enclosed by the curves $y = \sqrt{x}$, $y = 0$, and $y = x - 2$.
(a) First solution:



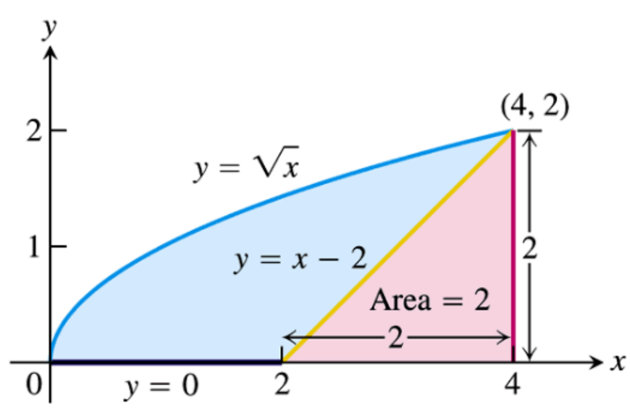
We have

$$\text{Area of } R = \text{Area of } A + \text{Area of } B.$$

We determine the right-hand limit for A by solving the simultaneous equations $y = 0$ and $y = x - 2$, this gives $x = 2$. We determine the right-hand limit for B by solving $y = \sqrt{x}$ and $y = x - 2$, this gives $x = 4$. Hence

$$\begin{aligned}\text{Area of } R &= \int_0^2 \sqrt{x} - 0dx + \int_2^4 \sqrt{x} - (x-2)dx \\ &= \frac{2}{3}x^{3/2}\Big|_0^2 + \left(\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right)\Big|_2^4 \\ &= \frac{10}{3}\end{aligned}$$

(b) Second solution:



The area below the curve $y = \sqrt{x}$ and above $[0, 4]$ is

$$A_1 = \int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{16}{3}.$$

The area of the triangle is $A_2 = 2 \cdot 2/2 = 2$. Hence

$$\text{Area of } R = A_1 - A_2 = \frac{16}{3} - 2 = \frac{10}{3} \ .$$

(c) Third solution: We consider the area to be a limit of Riemann sums of *horizontal rectangles* rather than vertical rectangles. To do this we need to express the equations of the curves $y = \sqrt{x}$ and $y = x - 2$ as functions with y as the dependent variable and x as the independent variable i.e. $x = y^2$ and $x = y + 2$. Then each horizontal rectangle will have length $(y + 2) - y^2$ and width Δy so the area of R is given by

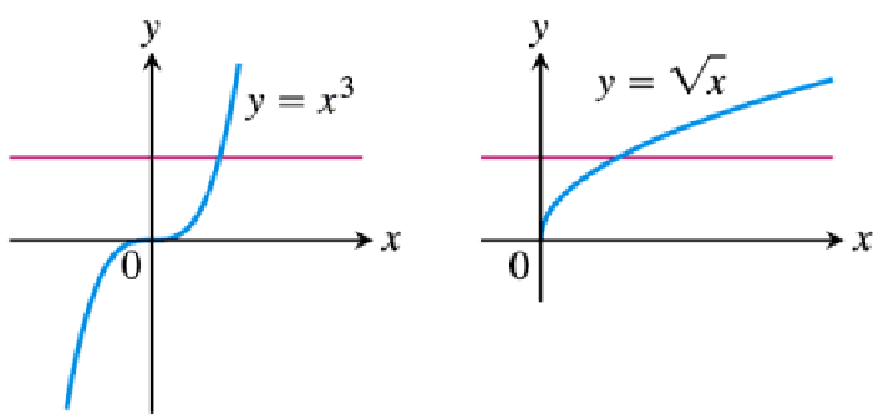
$$\int_0^2 y + 2 - y^2 \, dy = \left(\frac{y^2}{2} + 2y - \frac{y^3}{3} \right) \Big|_0^2 = \frac{10}{3}.$$

Inverse functions and their derivatives

Definition Let $f : D \to \mathbb{R}$ be a function. Then f is *injective* (or *one-to-one*) if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

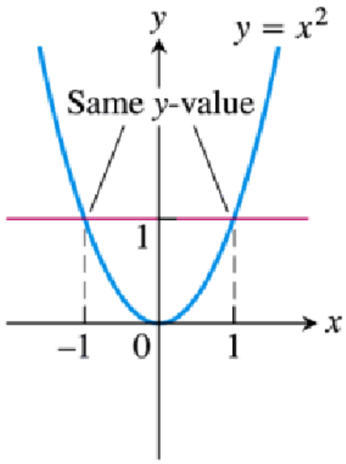
Thus a function is injective if it takes on every value in its range exactly once.

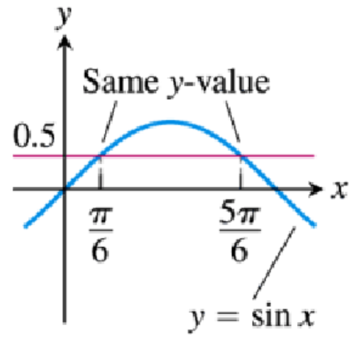
Examples: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$ and $g : \mathbb{R}^+ \to \mathbb{R}$ be defined by $g(x) = \sqrt{x}$, where $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. Then f and g are both injective.



The horizontal line test for injective functions: A function is injective if and only if its graph intersects every horizontal line at most once.

Examples: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ and $g : [0, \pi] \to \mathbb{R}$ be defined by $g(x) = \sin x$. Then neither f nor g is injective.





Note that if we restrict the domain of f to $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ and the domain of g to $[0, \pi/2]$ then the restricted functions will both be injective.

Definition Suppose that $f : D \to \mathbb{R}$ is an injective function with range R. Then the *inverse function* $f^{-1} : R \to D$ is defined by

$$f^{-1}(y) = x \text{ whenever } f(x) = y.$$

Note that:

- the domain of f^{-1} is equal to the range of f and the range of f^{-1} is equal to the domain of f.
- f^{-1} is read as f *inverse*.
- $f^{-1}(x) \neq 1/f(x)$ i.e. $f^{-1}(x)$ is not the same as $f(x)^{-1}$.
- the composition $f^{-1} \circ f$ maps each element of D onto itself i.e. $(f^{-1} \circ f)(x) = x$ for all $x \in D$.
- the composition $f \circ f^{-1}$ maps each element of R onto itself i.e. $(f \circ f^{-1})(y) = y$ for all $y \in R$.

Method for finding inverse functions:
Suppose that $f : D \to \mathbb{R}$ is an injective function with range R. When we write $y = f(x)$ we think of the dependent variable y as being a function of the independent variable x. To find f^{-1} we need to solve the equation $y = f(x)$ to express x as a function of y. This gives $x = f^{-1}(y)$ where x is now the dependent variable and y is the independent variable.
To obtain an expression for f^{-1} in the more standard form with y as the dependent variable and x is the independent variable we need to relabel x as y and y as x in the expression $x = f^{-1}(y)$.

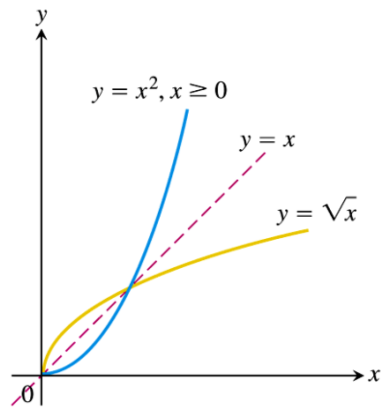
Example: Find the inverse of the function $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = x^2$.

Step 1 Solve $y = f(x)$ for x. We have $y = x^2$ and $x \geq 0$ so $x = \sqrt{y}$.
Since the domain and range of f is $\mathbb{R}^+$, we obtain

$$f^{-1} : \mathbb{R}^+ \to \mathbb{R}^+ \text{ by } f^{-1}(y) = \sqrt{y}$$

Step 2 Relabel y as x. This gives:

$$f^{-1} : \mathbb{R}^{+} \to \mathbb{R}^{+} \text{ by } f^{-1}(x) = \sqrt{x}$$



Relationship between graphs of f and f^{-1}
Recall that the graph of f is the set of all points $P = (a, b)$ satisfying $b = f(a)$. Since $b = f(a)$ if and only if $a = f^{-1}(b)$ we have:

$$P = (a, b) \text{ is on the graph of } f \text{ if and only if } Q = (b, a) \text{ is on the graph of } f^{-1}.$$

Since the points $P = (a, b)$ and $Q = (b, a)$ are interchanged by reflection in the line $y = x$ this implies:

the graphs of f and f^{-1} are interchanged by reflection in the line $y = x$.

Example See graphs of $f(x) = x^2$ and $f^{-1}(x) = \sqrt{x}$ above.

Derivatives of inverse functions

Theorem 2 *Suppose that $f : D \to \mathbb{R}$ is injective, differentiable and $f'(x) \neq 0$ for all $x \in D$. Then f^{-1} is differentiable and its derivative $(f^{-1})'$ satisfies*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} .$$

Equivalently, for all b in the domain of f^{-1} we have

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}} .$$

Proof Let $y = f^{-1}(x)$. Then $x = f(y)$. We can differentiate this second equation using the chain rule to obtain

$$1 = \frac{dx}{dx} = \frac{d}{dx} f(y) = f'(y) \frac{dy}{dx}$$

and hence

$$\frac{dy}{dx} = \frac{1}{f'(y)} .$$

Since $y = f^{-1}(x)$, this gives

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} .$$

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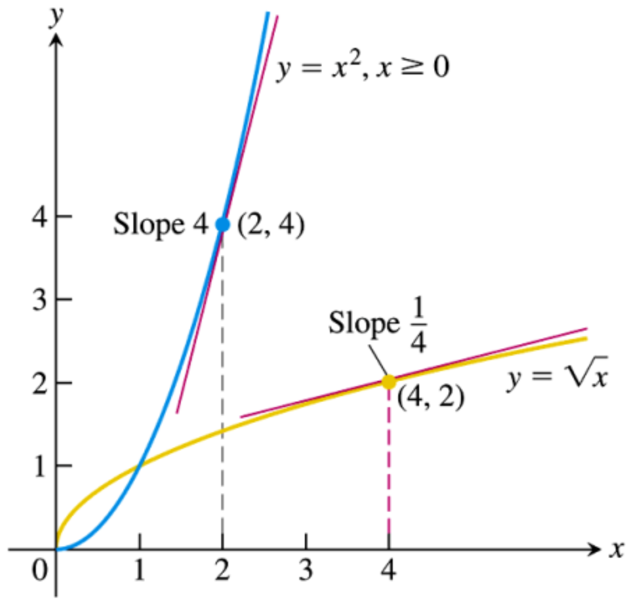
Example: Let $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = x^2$. Then $f^{-1}(x) = \sqrt{x}$. We have $f'(x) = 2x$ and $(f^{-1})'(x) = \frac{1}{2\sqrt{x}}$. Hence

$$\frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\sqrt{x})} = \frac{1}{2\sqrt{x}} = (f^{-1})'(x) .$$



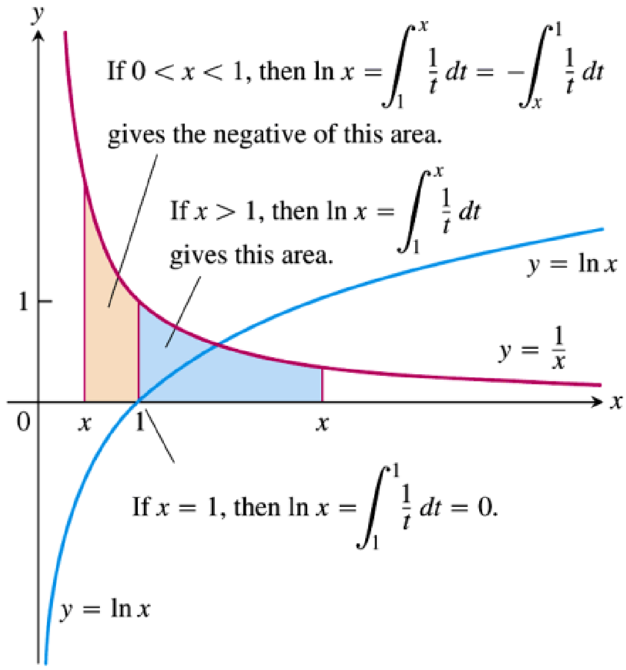
Note: The second part of the theorem can be used to find the value of the derivative of $f^{-1}(x)$ when $x = f(a) = b$ for any $a \in D$ without calculating the formula for f^{-1}.
Example continued: Suppose $f(x) = x^2$ and we want to determine the value of the derivative of $f^{-1}(x)$ when $x = f(2) = 4$. We have $f^{-1}(4) = 2$ so

$$\left.\frac{df^{-1}}{dx}\right|_{x=4} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}} = \frac{1}{\left.2x\right|_{x=2}} = \frac{1}{4} .$$

Natural Logarithms

Definition Consider the function $f(x) = x^{-1}$. This is continuous on the closed interval $[a, b]$ for all $0 < a < b$. The Fundamental Theorem of Calculus (Part 1) now tells us that $F(x) = \int_1^x t^{-1} dt$ is continuous on $[a, b]$ and differentiable on (a, b) for all $0 < a < b$. This function F is an important function: it is called the *natural logarithm function* and is denoted by ln. Thus

$$\ln x = \int_1^x t^{-1}\, dt .$$



Lemma 1 (Properties of the natural logarithm function) *The domain of* $\ln x$ *is* $(0, \infty)$ *and its derivative is* x^{-1}.

Proof The function $\ln x$ is defined for all $x > 0$ so its domain is $(0, \infty)$. The fact that the derivative of $\ln x$ is $1/x$ follows from the Fundamental Theorem of Calculus (Part 2):

$$\frac{d}{dx} \ln x = \frac{d}{dx} \int_1^x t^{-1} dt = x^{-1}\,.$$

•

Lemma 2 (Rules for manipulating natural logarithms) *Suppose* a, x *are positive real numbers. Then*

1. $\ln ax = \ln a + \ln x$.
2. $\ln \frac{1}{x} = -\ln x$.
3. $\ln \frac{a}{x} = \ln a - \ln x$.
4. $\ln x^q = q \ln x$ *for any rational number* q.

Proof of (1) By the chain rule

$$\frac{d}{dx} \ln ax = \frac{1}{ax}\frac{d}{dx} ax = \frac{1}{ax}\, a = \frac{1}{x} = \frac{d}{dx} \ln x$$

It follows that $\ln ax$ and $\ln x$ are both antiderivatives for $1/x$ and hence

$$\ln ax = \ln x + C$$

for some constant C. Substituting $x = 1$ we obtain

$$\ln a = \ln 1 + C = C$$

since $\ln 1 = \int_1^1 t^{-1}dt = 0$. Thus

$$\ln ax = \ln x + \ln a\,.$$

The proofs of rules (2)-(4) are similar, see Thomas page 372. •

Examples:

1. $\ln 8 + \ln\cos x = \ln(8\cos x)$
2. $\ln\dfrac{z^2+3}{2z-1} = \ln(z^2+3) - \ln(2z-1)$
3. $\ln\cot x = \ln\dfrac{1}{\tan x} = -\ln\tan x$
4. $\ln\sqrt[5]{x-3} = \ln(x-3)^{1/5} = \dfrac{1}{5}\ln(x-3)$

Lemma 3 (Range of the natural logarithm function) *The range of* $\ln x$ *is* $(-\infty,\infty)$.

Proof Since $1/x \geq 1/2$ for $x \in [1,2]$, the min-max rule for definite integrals tells us that

$$\ln 2 = \int_1^2 t^{-1}dt \geq (2-1)\frac{1}{2} = \frac{1}{2}\,.$$

We can now use Rule 4 for manipulating natural logarithms to deduce that $\ln 2^n = n\log 2 \geq n/2$ for any integer $n \geq 1$. Hence $\log 2^n$ becomes arbitrarily large and positive as n approaches infinity so $\lim_{n\to\infty}\log 2^n = \infty$. Since $\ln 2^{-n} = -\ln 2^n$, $\lim_{n\to\infty}\log 2^{-n} = -\infty$. The fact that $\ln x$ is continuous now implies that $\ln x$ takes all values in $(-\infty,\infty)$. •

Definition The fact that the range of $\ln x$ is $(-\infty,\infty)$ implies in particular that $\ln x = 1$ for some $x \in (0,\infty)$. The point e for which $\ln e = 1$ is referred to as *Euler's constant* or *the base of the natural logarithm*. Its approximate numerical value is $e = e = 2.718281828459\ldots$

We have seen that $\ln x$ is an antiderivative for $1/x$ for any interval $I \subset (0,\infty)$. Our next result extends this to all intervals which do not contain zero.

Theorem 3 *Let* I *be an interval. If* $0 \notin I$ *then* $\ln|x|$ *is an antiderivative for* $f(x) = 1/x$ *on* I*. More generally, if* $g(x)$ *is non-zero and differentiable on* I*, then* $\ln|g(x)|$ *is an antiderivative for* $g'(x)/g(x)$ *on* I*.*

Proof To show that $\ln|x|$ is an antiderivative for $f(x) = 1/x$ on I we need to show that $\frac{d}{dx}\ln|x| = 1/x$. We consider two cases.
Case 1: $I \subset (0,\infty)$. Then $\ln|x| = \ln x$ and $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = 1/x$.
Case 2: $I \subset (\infty,0)$. Then $\ln|x| = \ln(-x)$ and

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}\frac{d}{dx}(-x) = \frac{-1}{-x} = \frac{1}{x}$$

by the chain rule.

The second part of the lemma follows from the Substitution Law for Indefinite Integrals. We have seen that $F(x) = \ln|x|$ is an antiderivative for $1/x$. The substitution law now tells us that $F(g(x)) = \ln|g(x)|$ is an antiderivative for $g'(x)/g(x)$. •

Example For $x \in (-\pi/2, \pi/2)$ we have

$$\begin{aligned}
\int \tan x\, dx &= \int \frac{\sin x}{\cos x} dx \\
&= -\int \frac{1}{u}\, du \qquad \text{(Substitute } u = \cos x\text{, so } du = \sin x\text{)} \\
&= -\ln|u| + C \\
&= -\ln|\cos x| + C \\
&= \ln(1/|\cos x|) + C \\
&= \ln|\sec x| + C
\end{aligned}$$

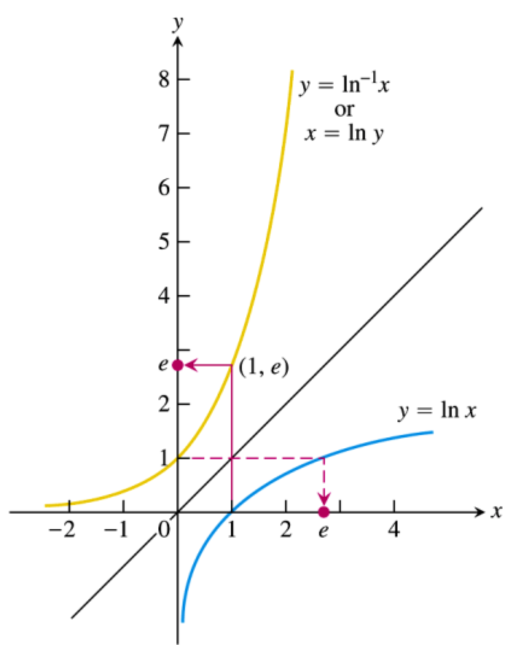
A similar calculation shows that

$$\int \cot x\, dx = \ln|\sin x| + C$$

for $x \in (0, \pi)$.

The Exponential Function

Definition The natural logarithm function $\ln x$ has domain $(0, \infty)$ and range $\mathbb{R}$. Since $\frac{d}{dx} \ln x = 1/x > 0$ on $(0, \infty)$, $\ln x$ is strictly increasing. This implies that $\ln x$ is injective and hence is invertible. Its inverse function $\exp(x) = \ln^{-1}(x)$ is another important function. It is called the *exponential function*.



Lemma 4 (Properties of the exponential function) *The domain of* $\exp x$ *is* $\mathbb{R}$ *and its range is* $(0, \infty)$. *The derivative of* $\exp x$ *is* $\exp x$.

Proof Since $\exp = \ln^{-1}$, the domain of $\exp x$ is equal to the range of $\ln x$, which is $\mathbb{R}$, and the range of $\exp x$ is equal to the domain of $\ln x$, which is $(0, \infty)$. The statement about the derivative of $\exp x$ follows from our general result on derivatives of inverse functions, but it is just as easy to calculate the derivative directly. Let $y = \exp x$. Then $x = \ln y$. Differentiating we get

$$1 = \frac{dx}{dx} = \frac{d}{dy}\ln y\,\frac{dy}{dx} = \frac{1}{y}\,\frac{dy}{dx}\,.$$

Hence $\frac{dy}{dx} = y$. Since $y = \exp x$ this gives $\frac{d}{dx}\exp x = \exp x$. •

Irrational powers of real numbers

We have defined what we mean by a^q for any real number $a > 0$ and any *rational* number q. We can use the exponential function to extend this to a definition of a^x when x is *irrational* i.e. $x \in \mathbb{R} \setminus \mathbb{Q}$. We first express a^q in terms of the exponential function.

Lemma 5 *Suppose a is a positive real number and $q \in \mathbb{Q}$. Then*

$$a^q = \exp(q \ln a)\,. \tag{1}$$

Proof The fourth rule for manipulating natural logarithms tells us that

$$\ln a^q = q \ln a\,.$$

Taking the exponential of both sides of this equation (and using $\exp = \ln^{-1}$) gives

$$a^q = \exp(\ln a^q) = \exp(q \ln a)\,.$$

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Since the right hand side of (1) makes sense for all $q \in \mathbb{R}$ we can use it define a^x for all real numbers x.

Definition For any $a \in \mathbb{R}$ with $a > 0$, the *exponential function with base a* is defined by putting

$$a^x = e^{x \ln a}$$

for all $x \in \mathbb{R}$.

Note that this definition implies that

$$\ln(a^x) = \ln[\exp(x \ln a)] = x \ln a \tag{2}$$

and hence that the fourth rule for manipulating natural logarithms holds for all powers of a, not just rational powers.

For the definition of a^x to make sense we will need the exponent in a^x to behave in the same way as exponents for integer or rational powers of a. This follows from our next result.

Lemma 6 *Suppose a is a positive real number and $b, c \in \mathbb{R}$. Then:*

1. $a^b \cdot a^c = a^{b+c}$ *;*

2. $(a^b)^c = a^{bc}$ *.*

Proof By definition $a^b = \exp(b \ln a)$ and $a^c = \exp(c \ln a)$. Hence

$$\begin{aligned} a^b \cdot a^c &= \exp[\ln(a^b \cdot a^c)] \\ &= \exp[\ln(a^b) + \ln(a^c)] \qquad \text{(by the first rule for manipulating logs)} \\ &= \exp[b\ln(a) + c\ln(a)] \qquad \text{(by 2)} \\ &= \exp[(b+c)\ln(a)] \\ &= a^{b+c}. \end{aligned}$$

Similarly

$$\begin{aligned} (a^b)^c &= \exp(c \ln a^b) \\ &= \exp(c \ln[\exp(b \ln a)]) \\ &= \exp(cb \ln(a)) \qquad (\text{since } \exp = \ln^{-1}) \\ &= a^{bc}. \end{aligned}$$

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Note: The exponential function with base a is differentiable for all $x \in \mathbb{R}$ and

$$\frac{d}{dx} a^x = \frac{d}{dx} \exp(x \ln a) = \exp(x \ln a) \cdot \ln a = a^x \ln a$$

by the chain rule. Hence

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

when $a > 0$ and $a \neq 1$.

Definition When $a > 1$, $\frac{d}{dx} a^x = a^x \ln a$ is positive and hence $f(x) = a^x$ is strictly increasing for all $x \in \mathbb{R}$. When $0 < a < 1$, a similar argument shows that $f(x) = a^x$ is strictly decreasing for all $x \in \mathbb{R}$. This implies that $f(x) = a^x$ is injective for all $x \in \mathbb{R}$ for any fixed $a > 0$ with $a \neq 1$. Hence its inverse function exists. This inverse function is called the *logarithm of x to the base a* and is denoted by $\log_a x$. We have

$$\log_a(a^x) = x = a^{\log_a x}$$

for all $x \in \mathbb{R}$. This gives

$$\ln x = \ln\left(a^{\log_a x}\right) = \log_a x \cdot \ln a\,.$$

and hence

$$\log_a x = \frac{\ln x}{\ln a}$$

Note: The algebra for $\log_a x$ is precisely the same as that for $\ln x$.

Further properties of the exponential function

The above definition of a^x gives us an *alternative notation* for $\exp(x)$. Recall that $1 = \ln e$ where e is Euler's constant. This implies that

$$e^x = \exp(x \ln e) = \exp x.$$

Henceforth we will often use e^x instead of $\exp x$.

We have seen that $\frac{d}{dx}e^x = e^x$. This gives

$$\int e^x dx = e^x + C\,.$$

We can now use the chain rule to deduce:

Lemma 7 *Let $f(x)$ be a differentiable function. Then*

$$\frac{d}{dx}e^{f(x)} = e^{f(x)} f'(x)$$

and

$$\int e^{f(x)} f'(x) dx = e^{f(x)} + C\,.$$

Examples:

1.
$$\frac{d}{dx}e^{\sin x} = e^{\sin x}\frac{d}{dx}\sin x = e^{\sin x}\cos x$$

2.
$$\begin{aligned}\int_0^{\ln 2} e^{3x} dx &= \int_0^{\ln 8} e^u \frac{1}{3} du \\ &= \left.\frac{1}{3}e^u\right|_0^{\ln 8} \\ &= \frac{7}{3}\end{aligned}$$

We defined e via $\ln e = 1$ and stated $e = 2.718281828459\ldots$.

Theorem 4 (The number e as a limit)

$$e = \lim_{x\to 0}(1+x)^{1/x}$$

Proof We have

$$\begin{aligned}\ln\left(\lim_{x\to 0}(1+x)^{1/x}\right) &= \lim_{x\to 0}\left(\ln(1+x)^{1/x}\right) \qquad \text{(continuity of } \ln x\text{)} \\ &= \lim_{x\to 0}\left(\frac{1}{x}\ln(1+x)\right) \qquad \text{(power rule)} \\ &= \lim_{x\to 0}\frac{1}{1+x} \qquad \text{(l'Hôpital)} \\ &= 1\end{aligned}$$

Taking exponentials of both sides gives:

$$\lim_{x\to 0}(1+x)^{1/x} = \exp\left[\ln\left(\lim_{x\to 0}(1+x)^{1/x}\right)\right] = \exp 1 = e\,.$$

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Read

Thomas' Calculus:

Section 7.7 Inverse trigonometric functions,

and Section 7.8, Hyperbolic functions

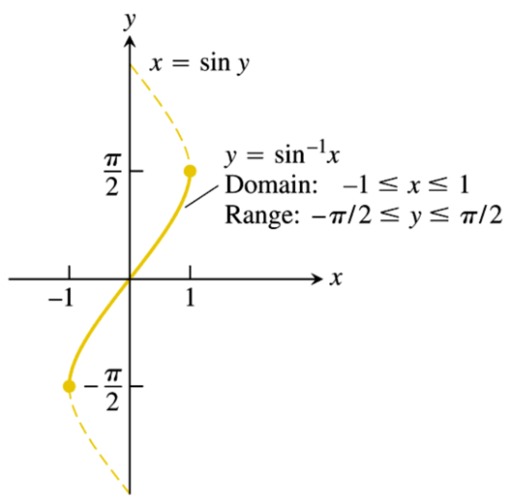
You will need this information for coursework 10!

The following two sections give a very brief summary of what can be found on these pages.

Inverse trigonometric functions

note: sin, cos, sec, csc, tan, cot are not injective *unless* the domain is restricted.

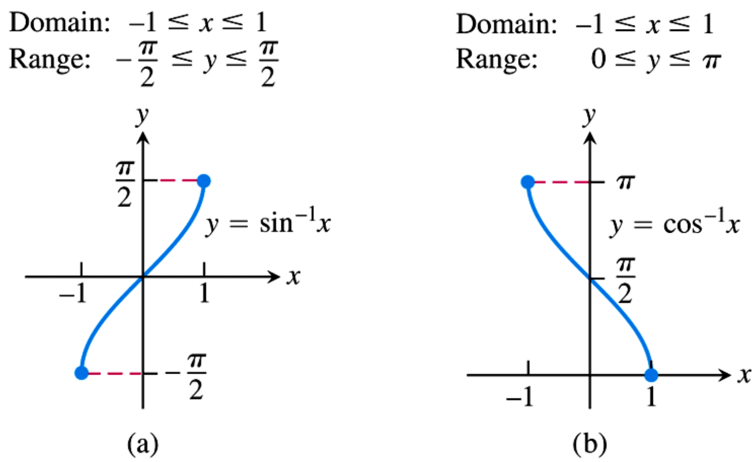
example:



Once the domains are suitably restricted, we can define:

$$\arcsin x = \sin^{-1} x \qquad \operatorname{arccsc} x = \csc^{-1} x$$
$$\arccos x = \cos^{-1} x \qquad \operatorname{arcsec} x = \sec^{-1} x$$
$$\arctan x = \tan^{-1} x \qquad \operatorname{arccot} x = \cot^{-1} x$$

examples:



...and so on.

caution: $$\sin^{-1} x \neq (\sin x)^{-1}$$

Unfortunately this is inconsistent, since $\sin^2 x = (\sin x)^2$. Best to avoid $\sin^{-1} x$ and use $\arcsin x$ etc. instead.

How to differentiate inverse trigonometric functions?

example: Differentiate $y = \arcsin x$.

Start with implicit differentiation of $\sin y = x$,

$$\cos y \frac{dy}{dx} = 1\ .$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$$

for $-\pi/2 < y < \pi/2$ ($\cos x = 0$ for $x = \pm\pi/2$). Therefore, for $|x| < 1$,

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

and, conversely,

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C\ .$$

example: Evaluate

$$\int \frac{dx}{\sqrt{4x-x^2}}\ .$$

Trick: complete the square!

$$4x - x^2 = 4 - (x-2)^2$$

Now integrate

$$\begin{aligned} \int \frac{dx}{\sqrt{4x-x^2}} &= \int \frac{dx}{\sqrt{4-(x-2)^2}} \\ (u = x-2) \quad &= \int \frac{du}{\sqrt{4-u^2}} \\ &= \arcsin \frac{u}{2} + C \\ &= \arcsin\left(\frac{x}{2} - 1\right) + C \end{aligned}$$

Hyperbolic functions

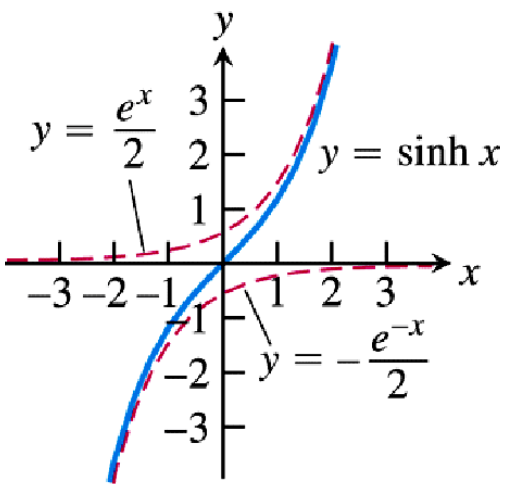
Every function f on $[-a, a]$ can be decomposed into

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even function}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd function}}$$

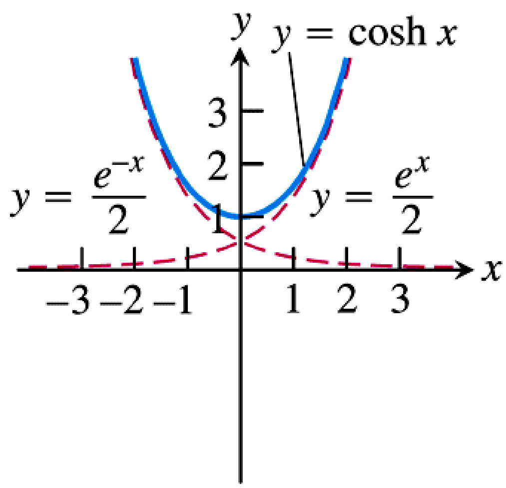
For $f(x) = e^x$:

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{=\cosh x} + \underbrace{\frac{e^x - e^{-x}}{2}}_{=\sinh x},$$

called *hyperbolic sine* and *hyperbolic cosine*.
Define tanh, coth, sech, and csch in analogy to trigonometric functions.
examples:



$$\sinh x = \frac{e^x - e^{-x}}{2}$$



$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Compare the following with trigonometric functions:

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

How to differentiate hyperbolic functions?
example:

$$\begin{aligned}\frac{d}{dx}\sinh x &= \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x \\ \frac{d}{dx}\cosh x &= \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x\end{aligned}$$

Inverse hyperbolic functions defined in analogy to trigonometric functions.