Queen Mary
University of London

MTH4101 Calculus II

Lecture notes for Week 11
Integration V and A First Look at Differential Equations

Thomas' Calculus, Sections 15.5, 15.8 and 7.4

Rainer Klages

School of Mathematical Sciences
Queen Mary, University of London

Spring 2013

Triple Integrals

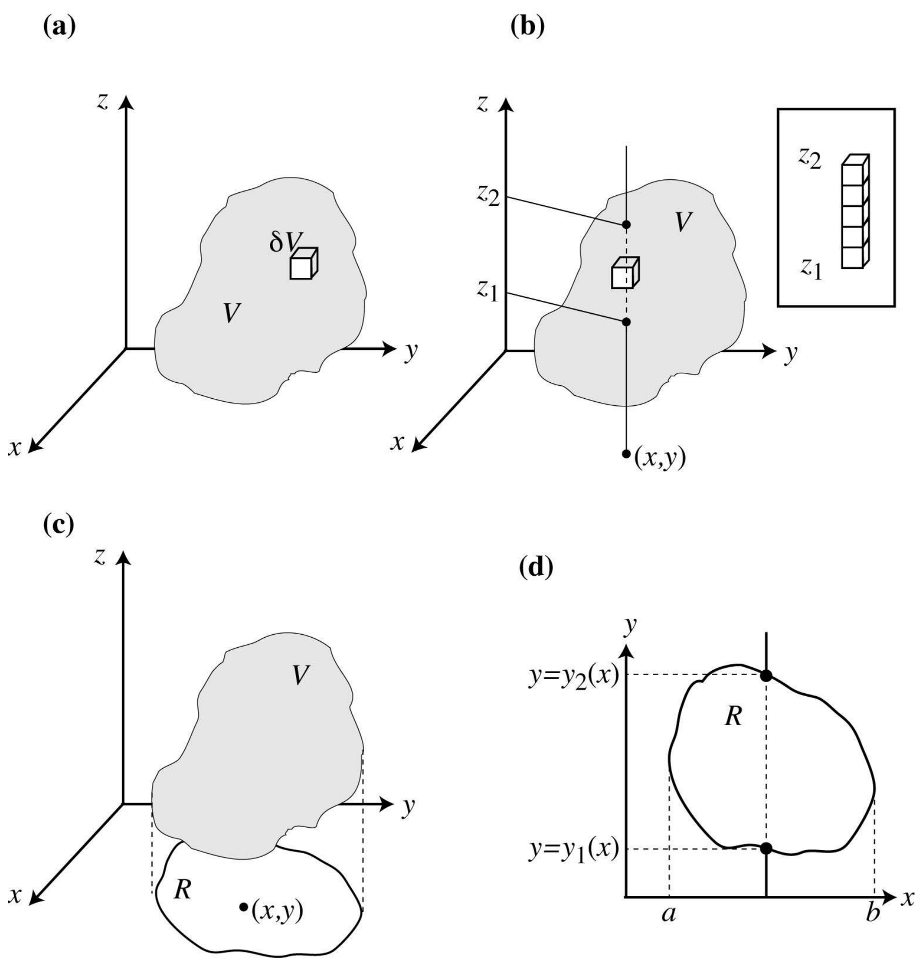
Triple integrals are integrations where the region of integration is a **volume**. The basic concepts are similar to those we introduced for two-dimensional (double) integrals, but now we have for the *Riemann sum*

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta V_k \ ,$$

where $\Delta V_k = \Delta x_k\, \Delta y_k\, \Delta z_k$ are now small volumes at the point x_k, y_k, z_k.
The limit as the size of the volume element $\Delta V_k \to 0$ (as $n \to \infty$) is written as (if it exists)

$$\lim_{n\to\infty} S_n = \int\!\!\int\!\!\int_V f(x,y,z)\, \mathrm{d}V = \int\!\!\int\!\!\int_V f(x,y,z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \ ,$$

where V is the three-dimensional region being integrated over.
The integrals are, as in the two-dimensional case, evaluated by repeated integration where we integrate over one variable at a time. For example, we could start by integrating over z first, see (b) in the figure below (where it is $\Delta V_k = \delta V$). The procedure is as follows:



1. **Sketch the region of integration** (if possible), see (a).
2. **Choose a direction of integration and integrate:** For example, fix a point (x, y) and integrate over the allowed values of z in the region V. The z-integral limits are the small, filled circles at the bottom and the top of the dashed line with, say, $z = z_1(x, y)$ at the bottom and $z = z_2(x, y)$ at the top as shown in (b). Therefore we are summing vertically over the boxes shown in (b).
3. This result depends on the choice of (x, y) and is defined in the region R of the (x, y) plane which is the projection of V onto this plane as shown in (c). This now **identifies the region in the (x, y) plane over which we must do the x and y integrations.**
4. Now we can **take the double integral** of the result of the z-integration **over the region R in the (x, y) plane**, see (d).

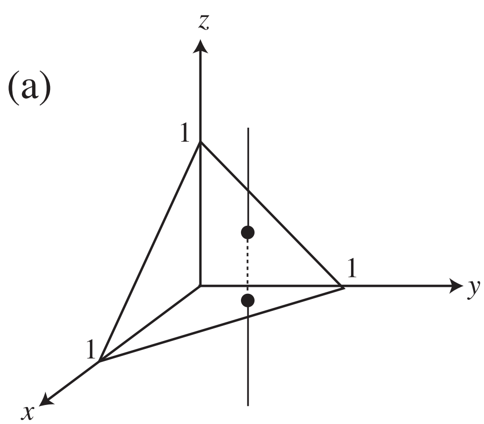
Therefore

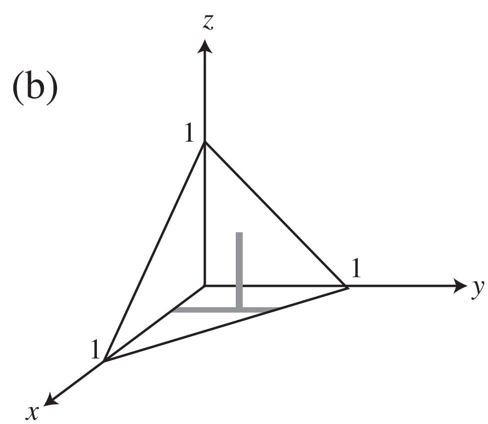
$$\int\int\int_V f(x, y, z)\, \mathrm{d}V = \int_{x=a}^{x=b} \int_{y=y_1(x)}^{y=y_2(x)} \int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x, y, z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x\,.$$

Example:
Evaluate

$$\int\int\int_T f(x, y, z)\, \mathrm{d}V$$

over the tetrahedron T bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. Note that the plane $x + y + z = 1$ passes through $x = 1$ (putting $y = z = 0$) and similarly through $y = 1$ and $z = 1$ as shown below:





Now evidently for fixed (x, y) the z-limits are the heavy dots corresponding to $z = 0$ at the bottom and $z = 1 - x - y$ at the top. This gives our z-limits.
The projection R of T onto the (x, y) plane is the triangle on which the tetrahedron rests, i.e. the triangle given by $x = 0$, $y = 0$ and $x + y = 1$ (obtained by setting $z = 0$). So

$$I = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} f(x, y, z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x\,.$$

For example, if $f(x, y, z) = 1$ then

$$I = \iiint_T 1 \cdot \mathrm{d}V = \iiint_T \mathrm{d}V = \textbf{volume of } T\,.$$

Therefore, in this case

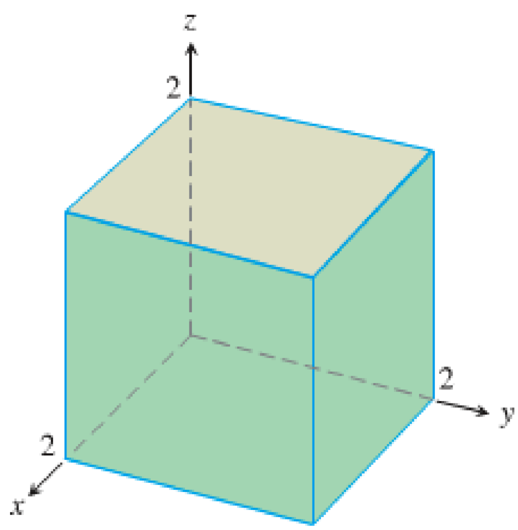
$$\begin{aligned} I &= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} 1\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} [z]_{z=0}^{z=1-x-y}\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} (1-x-y)\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{x=1}\left[y - xy - \frac{y^2}{2}\right]_{y=0}^{y=1-x}\mathrm{d}x \\ &= \int_{x=0}^{x=1}\frac{(1-x)^2}{2}\,\mathrm{d}x = \frac{1}{6} \end{aligned}$$

and this is the volume of the tetrahedron.

Triple integrals can be used to find the **average value of a function** $f(x,y,z)$ **over a volume** D defined as

$$\langle f(x,y,z)\rangle = \frac{1}{\text{volume of } D}\iiint_D f(x,y,z)\,\mathrm{d}V$$

Example:
Find the average value of $f(x,y,z) = xyz$ over the cube bounded by the planes $x = 2$, $y = 2$ and $z = 2$ in the first octant.



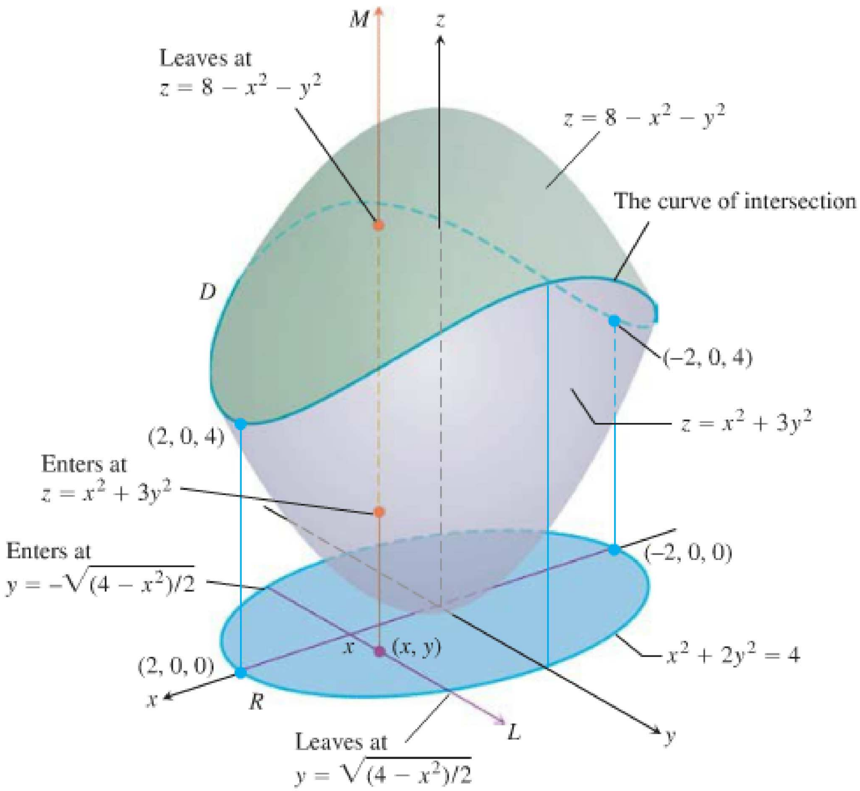
The volume of the cube is $2^3 = 8$. The integral is

$$\int_0^2\int_0^2\int_0^2 xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^2 x\,\mathrm{d}x\int_0^2 y\,\mathrm{d}y\int_0^2 z\,\mathrm{d}z = \left(\int_0^2 x\,\mathrm{d}x\right)^3 = \left(\left[\frac{x^2}{2}\right]_0^2\right)^3 = 8\,,$$

because the function is **separable** and the region is **cubic**. Therefore the average value of $f(x,y,z) = xyz$ over the cube is

$$\langle f(x,y,z)\rangle = \frac{1}{\text{volume of cube}}\iiint_{\text{cube}} xyz\,\mathrm{d}V = \frac{1}{8}\cdot 8 = 1\,.$$

Example:
Find the volume V of the region D enclosed by the surfaces $z = x^2+3y^2$ and $z = 8-x^2-y^2$.
The two surfaces intersect at $x^2 + 3y^2 = 8 - x^2 - y^2$. The equation $x^2 + 2y^2 = 4$ thus defines the boundary of the projection of D onto the x-y plane, which is the ellipse R:



We now have all the information necessary to do the integral:

$$
\begin{aligned}
V &= \iiint_D \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left(8-2x^2-4y^2\right)\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-2}^{2}\left[(8-2x^2)y - \frac{4}{3}y^3\right]_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\mathrm{d}x \\
&= \int_{-2}^{2}\left(2(8-2x^2)\sqrt{\frac{(4-x^2)}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right)\mathrm{d}x \\
&= \int_{-2}^{2}\left(8\left(\frac{4-x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right)\mathrm{d}x \\
&= \frac{4\sqrt{2}}{3}\int_{-2}^{2}\left(4-x^2\right)^{3/2}\mathrm{d}x \quad [\text{since } (8-8/3)/(2^{3/2}) = 4\sqrt{2}/3] \\
&= \frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} 4^{3/2}\left(\cos^2\theta\right)^{3/2}\cdot 2\,\cos\theta\,\mathrm{d}\theta \quad [\text{using subst. } x = 2\sin\theta]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4\sqrt{2}}{3}\cdot 16\int_{-\pi/2}^{\pi/2}\cos^4\theta\,\mathrm{d}\theta = \frac{4\sqrt{2}}{3}\cdot 16\int_{-\pi/2}^{\pi/2}\frac{1}{8}\left(3+4\cos 2\theta+\cos 4\theta\right)\,\mathrm{d}\theta \\
&= \frac{4\sqrt{2}}{3}\cdot 2\left[3\theta+2\sin 2\theta+\frac{1}{4}\sin 4\theta\right]_{-\pi/2}^{\pi/2} \\
&= \frac{4\sqrt{2}}{3}\cdot 2\cdot 3\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 8\sqrt{2}\,\pi\,.
\end{aligned}
$$

Substitution in Triple Integrals

Changing variables in triple integrals is similar to the procedure used for double integrals. Suppose

$$x = x(u,v,w), \qquad y = y(u,v,w), \qquad z = z(u,v,w)\,.$$

We define the **Jacobian matrix** for change of variables from (x, y, z) to (u, v, w) to be

$$\mathbf{M}(u,v,w) = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{pmatrix}\,.$$

and the corresponding **Jacobian determinant** as

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\mathbf{M}$$

such that the transformation for volume is

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w\,.$$

As before, for invertible transformations we have

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \left(\frac{\partial(u,v,w)}{\partial(x,y,z)}\right)^{-1}\,.$$

The integral under change of variables becomes

$$
\begin{aligned}
&\iiint_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \\
&\qquad \iiint_{V'} f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w\,,
\end{aligned}
$$

where V' is the transformed volume in (u, v, w) coordinates.

Example:
A volume V in the first octant is bounded by the six surfaces $xy = 1$, $xy = 2$, $yz = 1$, $yz = 2$, $xz = 1$ and $xz = 2$. Using the change of variables,

$$r = xy, \qquad s = yz, \qquad t = xz$$

evaluate the integral

$$\iiint_V xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

The new limits are $r = 1$ to $r = 2$, $s = 1$ to $s = 2$ and $t = 1$ to $t = 2$. The Jacobian determinant is

$$\begin{aligned}\frac{\partial(r,s,t)}{\partial(x,y,z)} &= \begin{vmatrix}\partial r/\partial x & \partial r/\partial y & \partial r/\partial z\\ \partial s/\partial x & \partial s/\partial y & \partial s/\partial z\\ \partial t/\partial x & \partial t/\partial y & \partial t/\partial z\end{vmatrix} = \begin{vmatrix} y & x & 0\\ 0 & z & y\\ z & 0 & x\end{vmatrix}\\ &= y\begin{vmatrix} z & y\\ 0 & x\end{vmatrix} - x\begin{vmatrix} 0 & y\\ z & x\end{vmatrix}\\ &= y(xz) + x(yz) = 2xyz\,.\end{aligned}$$

But

$$\frac{\partial(x,y,z)}{\partial(r,s,t)} = \left(\frac{\partial(r,s,t)}{\partial(x,y,z)}\right)^{-1} = \frac{1}{2xyz}$$

and so

$$\begin{aligned}\iiint_V xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \iiint_{V'} xyz\left|\frac{1}{2xyz}\right|\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t = \int_{t=1}^{t=2}\int_{s=1}^{s=2}\int_{r=1}^{r=2}\frac{1}{2}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t\\ &= \frac{1}{2}\,[r]_1^2\,[s]_1^2\,[t]_1^2 = \frac{1}{2}\cdot 1\cdot 1\cdot 1 = \frac{1}{2}\,.\end{aligned}$$

First-order differential equations and their solutions

You have learned in Calculus 1 that a function y is an **antiderivative** of a function f if

$$\frac{dy}{dx} = f(x)\,.$$

Finding an antiderivative for a given function $f(x)$ means finding a function $y(x)$ that solves this equation. This is an example of a **differential equation**, an equation involving the derivative of an unknown function y.

Using $f = f(x)$ on the right hand side the above equation defines a special case of a differential equation, and you already know of how to solve it. More generally, a **first-order differential equation** is of the form

$$\frac{dy}{dx} = f(x,y)\,,$$

where $f = f(x,y)$ is a function of *both* the independent variable x and the dependent variable y defined on a region in the xy-plane. The equation is of *first-order*, because it involves only the first derivative dy/dx (and not higher-order derivatives).

A **solution** of this equation is a differentiable function $y = y(x)$ defined on an interval I of x-values such that

$$\frac{d}{dx}y(x) = f(x, y(x))$$

on I. The **general solution** to such an equation is a solution that contains all possible solutions. As you will see in a moment (recall solving an indefinite integral), it always contains an arbitrary (integration) constant. This constant can be fixed by specifying an **initial condition**

$$y(x_0) = y_0\,.$$

The combination of a differential equation and an initial condition is called an **initial value problem**. The solution satisfying the initial condition $y(x_0) = y_0$ is the **particular solution** $y = y(x)$ whose graph passes through the point (x_0, y_0) in the xy-plane.

Example:
Show that

$$y = (x+1) - \frac{1}{3}e^x$$

solves the first-order initial value problem

$$\frac{dy}{dx} = y - x \ , \ y(0) = \frac{2}{3} \ .$$

Differentiate $y(x)$ to calculate the left hand side:

$$\frac{dy}{dx} = 1 - \frac{1}{3}e^x \ .$$

Now check for the right hand side:

$$y - x = (x+1) - \frac{1}{3}e^x - x = 1 - \frac{1}{3}e^x \ .$$

Both are equal, hence y solves the given equation. Since

$$y(0) = 1 - \frac{1}{3} = \frac{2}{3}$$

it also satisifies the initial condition.

Separable differential equations

An important class of first-order differential equation can be motivated by an

Example:
Solve the first-order differential equation.

$$\frac{dy}{dx} = ky \ ,$$

where the function $f(y) = ky$ on the right hand side only depends on y and is furthermore *linear* in y with a constant $k \in \mathbb{R}$.

By assuming that $y \neq 0$ we can write

$$\frac{1}{y}\frac{dy}{dx} = k \ .$$

If we treat dy/dx as a quotient of differentials dy and dx (by which strictly speaking we modify the problem - it defines a derivative!), we obtain

$$\frac{1}{y}dy = kdx$$

Now we can integrate:

$$\begin{aligned} \int \frac{1}{y} dy &= \int k dx \\ \ln|y| &= kx + C \, , \; C = \text{const.} \\ |y| &= e^{kx} e^{C} \\ y &= Ae^{kx} \text{ with } A = \pm e^{C} \, . \end{aligned}$$

We see that the solution of this differential equation undergoes *exponential change.*

The above example is a special case of what is called a **separable differential equation** $y' = f(x, y)$, where f can be expressed as a product of a function of x and a function of y. We can always try to solve such an equation by **separation of variables**:

$$\begin{aligned} y' &= g(x)h(y) \\ \frac{1}{h(y)} y' &= g(x) \end{aligned}$$

The detailed justification of what we have done in the previous example is integration by substitution

$$\int \frac{1}{h(y)} y' dx = \int g(x) dx$$

using $u = y(x)$,

$$\int \frac{1}{h(y)} dy = \int g(x) dx \, .$$

After completing the integrations on both sides (which may not always be possible), we obtain the solution y as a function of x in *implicit form.*