MTH4100 Calculus I

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- Choose a point $c_k \in [x_{k-1}, x_k]$.
- The sum $\sum_{k=1}^{n} f(c_k)\Delta x_k$ is called the *Riemann sum for f on $[a, b]$ with respect to the partition P and the choice of the points c_k*.

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- The sum $\sum_{k=1}^{n} f(c_k)\Delta x_k$ is called the *Riemann sum for f on $[a, b]$ with respect to the partition P and the choice of the points c_k*.
- The special cases when we choose c_k to be the maximum or minimum value of f on $[x_{k-1}, x_k]$ are called the *upper* and *lower sums*, respectively.

The indefinite integral

Definition Let f be a function defined on a closed interval $[a, b]$. A real number J is the *definite integral of f over* $[a, b]$ if J is the limit of all possible Riemann sums for f on $[a, b]$ as the width of the largest subinterval in the partitions goes to zero. If such a number J exists we write

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To decide if f is integrable over $[a, b]$ it suffices to show that the upper sums and the lower sums have the same limit (since all other Riemann sums are sandwiched between these two sums).

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If f is discontinuous then it may not be integrable.
Example:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

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Suppose that f *and* g *are integrable functions on* $[a, b]$. *Then:*

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(f) *If m is the absolute minimum of f on* $[a, b]$ *and M is the absolute maximum of f on* $[a, b]$ *then* $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$ *(max-min inequality);*

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(g) *If* $f(x) \leq g(x)$ *for all* $x \in [a, b]$ *then* $\int_a^b f(x)dx \leq \int_a^b g(x)dx$ *(domination).*

DEFINITION The Average or Mean Value of a Function

If f is integrable on $[a, b]$, then its **average value on** $[a, b]$, also called its **mean value**, is

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Example: $f(x) = 1 - x^2$, $x \in [0, 1]$.

Theorem (The mean value theorem for definite integrals)

Suppose f is continuous on $[a, b]$. Then there is a $c \in [a, b]$ with

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Corollary

Suppose f is a continuous function on a closed interval $[a, b]$ with $a \neq b$ and $\int_a^b f(x)dx = 0$. Then there is a $c \in [a, b]$ with $f(c) = 0$.

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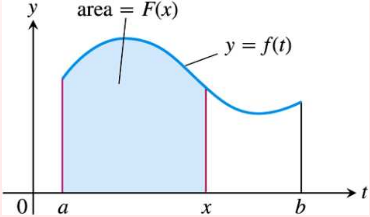
for all $x \in [a, b]$.



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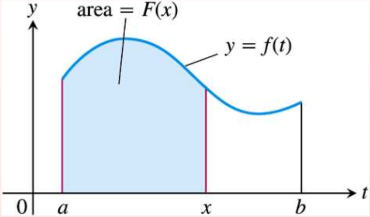

Example $f(x) = x$ on $[0, b]$ for some $b > 0$.

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Theorem (Fundamental Theorem of Calculus - Part 1)

Let f be a continuous function on a closed interval $[a, b]$. Then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b). Furthermore, the derivative of $F(x)$ is $f(x)$ i.e.

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Examples: Find

1. $$\frac{d}{dx}\int_a^x \frac{1}{1+4t^3}\,dt$$

2. $$\frac{d}{dx}\int_2^{x^2} \cos t\,dt$$

Fundamental Theorem of Calculus - Part 2

Part 2 tells us how to use antiderivatives to calculate definite integrals.

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Theorem (Fundamental Theorem of Calculus - Part 2)

Let f be a continuous function on a closed interval $[a, b]$ and F be ANY antiderivative for f. Then

$$\int_a^b f(t)dt = F(b) - F(a).$$

Calculating definite integrals

Method to evaluate $\int_a^b f(x)dx$

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Example: Find

$$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx$$

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We first consider indefinite integrals. Recall that the indefinite integral $\int f(x)dx$ is the general antiderivative for $f(x)$, and that it has the form $F(x) + C$ where $F'(x) = f(x)$ and C is an arbitrary constant.

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Example Evaluate

$$\int \frac{2z}{\sqrt[3]{z^2+5}}dz$$

Theorem (Substitution Rule for Indefinite Integrals)

Suppose g is a differentiable function and f is continuous function on the range of g. Let $F(x)$ be an antiderivative for $f(x)$. Then $F(g(x))$ is an antiderivative for $f(g(x))g'(x)$. Equivalently, if we put $u = g(x)$, then we have

$$\int f(g(x))g'(x)dx = \int f(u)du\,.$$

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Method for evaluating

$$\int f(g(x))g'(x)dx$$

1. Substitute $u = g(x)$, $du = g'(x)dx$ to obtain $\int f(u)du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

Theorem (Substitution Rule for Definite Integrals)

Suppose g is a differentiable function, g' is continuous on $[a, b]$, and f is continuous on the range of g. Then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du\,.$$

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Example: Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx$.