Queen Mary
University of London

MTH4100 Calculus I

Lecture notes for Week 10

Thomas' Calculus, Sections 5.2 to 5.5

Prof Bill Jackson

School of Mathematical Sciences
Queen Mary University of London

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Integrable functions

Definition Let f be a function defined on a closed interval $[a, b]$. We say that f *is integrable over* $[a, b]$ if the definite integral $\int_a^b f(x)dx$ exists.

The definition of the definite integral is complicated and it is not clear from this definition when a function will be integrable. Our next result gives us a large family of integrable functions.

Theorem 1 (Existence of Definite Integrals) *Suppose that f is a continuous function on a closed interval $[a, b]$. Then f is integrable over $[a, b]$.*

Proof Idea Show that the Upper and Lower Riemann sums have the same limit, see Thomas' page 264.

It is not true that all functions are integrable. Consider the following discontinuous function f defined on $[0, 1]$:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Since every subinterval of $[0, 1]$ of width greater than zero contains both a point in $\mathbb{Q}$ and a point in $\mathbb{R} \setminus \mathbb{Q}$, every upper sum is equal to 1, and every lower sum is equal 0. It follows that the limit of every sequence of upper sums will be 1, and the limit of every sequence of lower sums will be 0. Since the limits of these sums are not equal, $\int_0^1 f(x)dx$ does not exist.

Theorem 2 (Rules for definite integrals) *Suppose that f and g are integrable functions on a closed interval $[a, b]$. Then:*

(a) $\int_b^a f(x)dx = -\int_a^b f(x)dx$ *(reversing limits of integration);*

(b) $\int_a^a f(x)dx = 0$ *(area over a point is zero);*

(c) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ *for any constant* $k \in \mathbb{R}$ *(area scales by a constant multiplier);*

(d) $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ *(areas add);*

(e) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ *for any* $c \in [a, b]$ *(areas add);*

(f) If m is the global minimum of f on $[a, b]$ and M is the global maximum of f on $[a, b]$ then $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$ *(max-min inequality);*

(g) If $f(x) \leq g(x)$ *for all* $x \in [a, b]$ *then* $\int_a^b f(x)dx \leq \int_a^b g(x)dx$ *(domination).*

Rules (a) and (b) are definitions. Rules (c) to (g) can be proved using Riemann sums, see for example Thomas page 267 for a proof of (f).

The average value of a function

Suppose f is a function which is integrable on a closed interval $[a, b]$. Let P be a partition of $[a, b]$ into n subintervals of equal width. Then the width of every subinterval in P is $\Delta x = \frac{b-a}{n}$. Choose a point c_k in the k'th subinterval for each k, $1 \le k \le n$. Then the average value taken by f on the points $c_1, c_2, \ldots, c_n$ is

$$\frac{1}{n}\sum_{k=1}^{n} f(c_k) = \frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\frac{b-a}{n} = \frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\Delta x$$

Since f is integrable

$$\lim_{n\to\infty}\frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx\,.$$

It makes sense to consider this value to be the average value of f on $[a, b]$

DEFINITION The Average or Mean Value of a Function

If f is integrable on $[a, b]$, then its **average value on** $[a, b]$, also called its **mean value**, is

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

It follows that

$$\operatorname{av}(f)\,(b-a) = \int_a^b f(x)dx\,.$$

This tells us that the area of the rectangle of height $\operatorname{av}(f)$ and width $b-a$ is equal to the area under the graph of $y = f(x)$ over $[a, b]$.

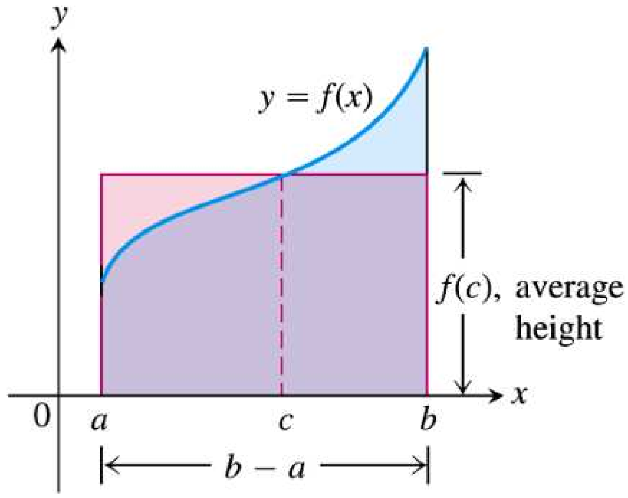
Example: $f(x) = 1 - x^2$, $x \in [0, 1]$. We have seen that the limits of the upper and lower Riemann sums are both equal to $\frac{2}{3}$ (see lecture notes for week 9). Hence $\int_0^1 (1 - x^2)dx = \frac{2}{3}$. This gives

$$\operatorname{av}(f) = \frac{1}{1-0}\int_0^1 x dx = \frac{2}{3}$$

Our next result shows that a continuous function on a closed interval will always take its average value at some point in the interval.

Theorem 3 (The mean value theorem for definite integrals) *Suppose f is continuous on $[a, b]$. Then there is a point $c \in [a, b]$ with*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx = av(f)\,.$$



Proof Let m, M be the absolute minimum and maximum values of f on $[a, b]$. The max-min-inequality for integrals tells us that

$$m \leq \frac{1}{b-a} \int_a^b f(x)dx \leq M\,.$$

The Intermediate Value Theorem for continuous functions now implies that there exists a $c \in [a, b]$ with

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx = \mathrm{av}(f)\,.$$

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Example continued: $f(x) = 1 - x^2\,,\ x \in [0, 1]$. To find c we need to solve $f(c) = \mathrm{av}(f)$ i.e. $1 - c^2 = 2/3$. This gives $c = 1/\sqrt{3}$.

Corollary 1 *Suppose f is a continuous function on a closed interval $[a, b]$ with $a \neq b$ and $\int_a^b f(x)dx = 0$. Then there is a $c \in [a, b]$ with $f(c) = 0$.*

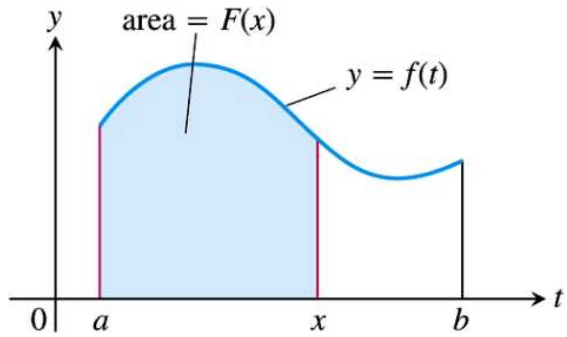
Proof Follows immediately from the Mean Value Theorem for Definite Integrals: this tells us there exists a $c \in [a, b]$ with $f(c) = \frac{1}{b-a} \int_a^b f(x)dx = 0$.

The Fundamental Theorem of Calculus

In this section, we will see how we can use antiderivatives to calculate definite integrals. Given a continuous function f on a closed interval $[a, b]$, define a new function F by putting

$$F(x) = \int_a^x f(t)dt$$

for all $x \in [a, b]$. Note that $F(x)$ exists for all $x \in [a, b]$ by the Existence Theorem for Definite Integrals.



Example $f(x) = x$ on $[0, b]$ for some $b > 0$. For any $x \in [0, b]$, $\int_0^x tdt$ is equal to the area of the triangle with base $[0, x]$ and height x. Hence $F(x) = \int_0^x tdt = x^2/2$.
In this example, $F(x) = x^2/2$ is an antiderivative of $f(x) = x$. Our next result tells us that this will always be true.

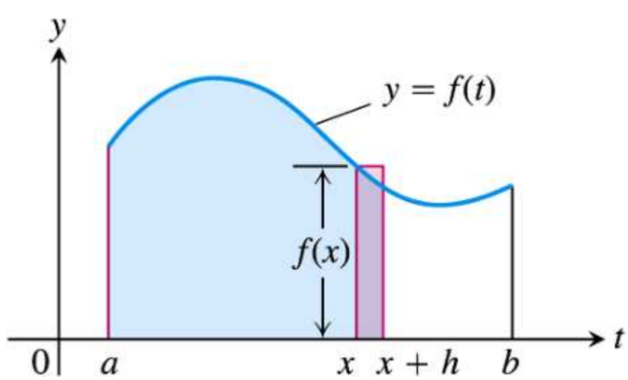
Theorem 4 (Fundamental Theorem of Calculus - Part 1) *Let f be a continuous function on a closed interval $[a, b]$. Then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b). Furthermore, the derivative of $F(x)$ is $f(x)$ i.e.*

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Proof We have

$$\begin{aligned}
\frac{F(x+h) - F(x)}{h} &= \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right) \\
&= \frac{1}{h}\int_x^{x+h} f(t)dt \qquad \text{(additivity rule)} \\
&= f(c)
\end{aligned}$$

for some $c \in [x, x+h]$ by the Mean Value Theorem for Definite Integrals.



Since f is continuous and $c \in [x, x+h]$, $f(c)$ will become arbitrarily close to $f(x)$ as h approaches zero. Hence

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)\,.$$

This implies that F is differentiable at x and $F'(x) = f(x)$ for all $x \in (a, b)$. Thus F is differentiable (and hence also continuous) on (a, b). To show F is continuous on the closed interval $[a, b]$ we need to show that $\lim_{x\to a^+} F(x) = F(a)$ and $\lim_{x\to b^-} F(x) = F(b)$. We leave this as an Exercise. •

Examples:

1.
$$\frac{d}{dx}\int_a^x \frac{1}{1+4t^3}dt = \frac{1}{1+4x^3}$$

2. Find
$$\frac{d}{dx}\int_2^{x^2} \cos t\, dt \;:$$

   Define
$$y = \int_2^u \cos t\, dt \text{ with } u = x^2$$

   Apply the chain rule:
$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\ &= \left(\frac{d}{du}\int_2^u \cos t\, dt\right)\cdot\frac{du}{dx} \\ &= \cos u \cdot 2x \\ &= 2x\cos x^2 \end{aligned}$$

We next show that we can calculate the definite integral $\int_a^b f(x)dx$ by evaluating $F(b)-F(a)$ for any antiderivative F of f.

Theorem 5 (Fundamental Theorem of Calculus - Part 2) *Let f be a continuous function on a closed interval $[a, b]$ and F be ANY antiderivative for f. Then*

$$\int_a^b f(t)dt = F(b) - F(a)\,.$$

Proof Let $G(x) = \int_a^x f(t)dt$. We have just shown that $G'(x) = f(x)$ and hence G is an antiderivative of f. Suppose that F is any other antiderivative of f. Then $F(x) = G(x)+C$ for some constant C. Hence

$$\begin{aligned} F(b) - F(a) &= (G(b) + C) - (G(a) + C) \\ &= G(b) - G(a) \\ &= \int_a^b f(t)dt - \int_a^a f(t)dt \\ &= \int_a^b f(t)dt \end{aligned}$$

since $\int_a^a f(t)dt = 0$ by the zero width interval rule.

Method to evaluate $\int_a^b f(x)dx$

Step 1 Find an antiderivative F of f.

Step 2 Calculate $F(b) - F(a)$.

Notation: Let $F(b) - F(a) = F(x)\big|_a^b$.

Example:

$$\begin{aligned} \int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx &= \left(x^{3/2} + \frac{4}{x}\right)\Big|_1^4 \\ &= \left(4^{3/2} + \frac{4}{4}\right) - \left(1^{3/2} + \frac{4}{1}\right) \\ &= 4 \end{aligned}$$

Fundamental Theorem of Calculus: Summary

Suppose that f is a continuous function on $[a, b]$. Then
(a)

$$\frac{d}{dx}\int_a^b f(t)\,dt = f(x)$$

for all $x \in [a, b]$ so $F(x) = \int_a^b f(t)\,dt$ is an antiderivative for f on $[a, b]$.
(b) If F is any antiderivative for f on $[a, b]$ then

$$\int_a^x \frac{dF}{dt}dt = \int_a^x f(t)dt = F(x) - F(a)\,.$$

This theorem tells us in particular that the operations of integration and differentiation are "inverses" of each other.

The substitution rule

This rule gives us a technique for calculating certain integrals. We will look at other techniques later.
We first consider indefinite integrals. Recall that the indefinite integral $\int f(x)dx$ is the general antiderivative for $f(x)$, and that it has the form $F(x) + C$ where $F'(x) = f(x)$ and C is an arbitrary constant.

Example Evaluate

$$\int \frac{2z}{\sqrt[3]{z^2+5}}dz$$

1. Substitute $u = z^2 + 5$, $du = 2z\,dz$:

$$\int \frac{2z}{\sqrt[3]{z^2+5}}dz = \int u^{-1/3}du$$

2. Integrate:

$$\int u^{-1/3} du = \frac{3}{2} u^{2/3} + C$$

3. Replace u by $z^2 + 5$:

$$\int \frac{2z}{\sqrt[3]{z^2+5}} dz = \frac{3}{2}(z^2+5)^{2/3} + C$$

In general we have

Theorem 6 (Substitution Rule for Indefinite Integrals) *Suppose g is a differentiable function and f is continuous function on the range of g. Let $F(x)$ be an antiderivative for $f(x)$. Then $F(g(x))$ is an antiderivative for $f(g(x))g'(x)$. Equivalently, if we put $u = g(x)$, then we have*

$$\int f(g(x))g'(x)dx = \int f(u)du\,.$$

Proof Let $F(u) + C$ be the general antiderivative for $f(u)$ i.e. $\int f(u)du = F(u) + C$. We need to show that $F(g(x)) = F(u)$ is an antiderivative for $f(g(x))g'(x)$. This follows from the chain rule since

$$\frac{d}{dx}F(u) = \frac{d}{du}F(u).\frac{du}{dx} = f(u).\frac{du}{dx} = f(g(x))g'(x)\,.$$

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Method for evaluating

$$\int f(g(x))g'(x)dx \ :$$

1. Substitute $u = g(x)$, $du = g'(x)dx$ to obtain $\int f(u)du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

We next consider definite integrals.

Theorem 7 (Substitution Rule for Definite Integrals) *Suppose g is a differentiable function, g' is continuous on $[a, b]$, and f is continuous on the range of g. Then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du\,.$$

Proof Let F be an antiderivative for f. The substitution rule for indefinite integrals tells us that $F(g(x))$ is an antiderivative for $f(g(x))g'(x)$. We can now use the Fundamental Theorem of Calculus to deduce that

$$\int_a^b f(g(x))g'(x)dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)du\,.$$

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Example: Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx$.

Substitute $u = x^3 + 1$, $du = 3x^2 dx$.
$x = -1$ gives $u = (-1)^3 + 1 = 0$; $x = 1$ gives $u = 1^3 + 1 = 2$, and we obtain

$$\begin{aligned}
\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx &= \int_0^2 \sqrt{u}du \\
&= \frac{2}{3}u^{3/2}\bigg|_0^2 \\
&= \frac{2}{3}2^{3/2} - 0 \\
&= \frac{4\sqrt{2}}{3}
\end{aligned}$$