MTH4101 Calculus II

Lecture notes for Week 10
Integration V

Thomas' Calculus, Sections 15.8, 15.4 and 15.5

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Substitution in Double Integrals

For functions of one variable it is often useful to integrate by a change of variable, e.g. $x = x(u)$. The rule is to replace x by $x(u)$ and $\mathrm{d}x$ by $(\mathrm{d}x/\mathrm{d}u)\mathrm{d}u$ and then alter the x-limits to the u-limits. This is **integration by substitution**, which gives

$$I = \int_{x=a}^{x=b} f(x)\,\mathrm{d}x = \int_{u=u_1}^{u=u_2} f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u\ ,$$

where u_1 and u_2 correspond to the limits a and b such that $a = x(u_1)$ and $b = x(u_2)$.
The above equation follows straightforwardly if $x(u)$ *increases* with u. If $x(u)$ is a *decreasing* function of u the u-limits are reversed and therefore we have a change of sign:

$$I = \int_{x=a}^{x=b} f(x)\,\mathrm{d}x = -\int_{u=u_1}^{u=u_2} f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u\,.$$

But $\mathrm{d}x/\mathrm{d}u < 0$ in this case, so we can combine both cases in one formula:

$$\int_{x=a}^{x=b} f(x)\,\mathrm{d}x = \int_{u=u_1}^{u=u_2} f(x(u))\left|\frac{\mathrm{d}x}{\mathrm{d}u}\right|\,\mathrm{d}u\,.$$

Note that on the right-hand side of this equation the function $f(x)$ is expressed as $f(x(u))$. Also, the right-hand side of the equation includes a *scaling factor* $|\mathrm{d}x/\mathrm{d}u|$, multiplying the $\mathrm{d}u$; this comes from transforming from $\mathrm{d}x$ to $\mathrm{d}u$.
For functions of two variables one would similarly expect that the change in variables

$$x = x(u,v), \quad y = y(u,v)$$

(for example, for polar coordinates $u = r$ and $v = \theta$) would result in a change in the area by a *scaling factor* S such that

$$\mathrm{d}x\,\mathrm{d}y = S\,\mathrm{d}u\,\mathrm{d}v\,.$$

As an example consider a *linear change* of coordinates:

$$x = x(u,v) = au + bv, \qquad y = y(u,v) = cu + dv$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

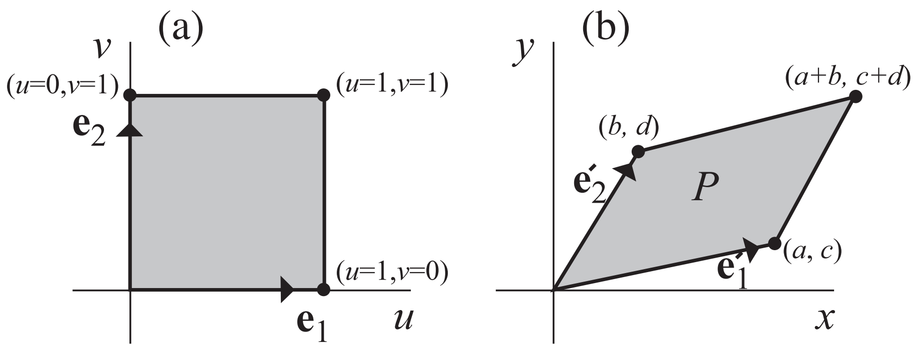
where a, b, c and d are constants.
Let us write $\mathbf{M}$ for the transformation matrix composed of a, b, c and d and recall that a unit square in (u,v) variables has sides

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{e}_1, \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{e}_2$$

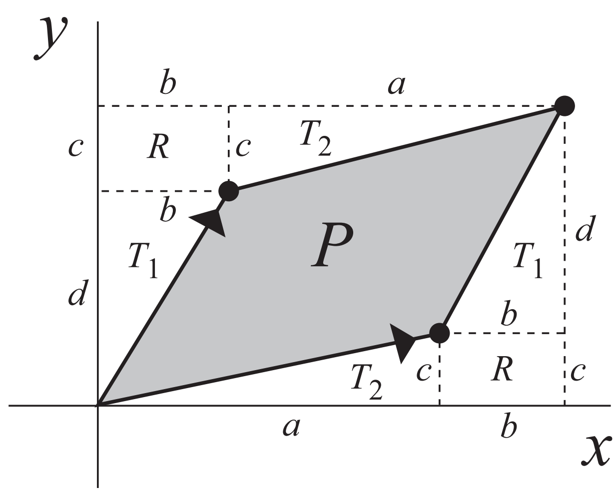
To see what happens to this unit square under the transformation $\mathbf{M}$, just apply $\mathbf{M}$. This gives

$$\begin{aligned} \mathbf{M}\,\mathbf{e}_1 &= \mathbf{e}_1' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \\ \mathbf{M}\,\mathbf{e}_2 &= \mathbf{e}_2' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} \end{aligned}$$

where (a,c) and (b,d) represent the coordinates of the new corners in the (x,y) plane:



Therefore, under the transformation $\mathbf{M}$ we find that the unit square in (u, v) based on $\mathbf{e}_1$, $\mathbf{e}_2$ is transformed into the parallelogram in (x, y) based on $\mathbf{e}'_1$, $\mathbf{e}'_2$.
Note from the matrix and the diagram that the point $(1, 1)$ in (u, v) transforms to the point $(a + b, c + d)$ in (x, y).
Let us calculate the area of the parallelogram P:



We have

$$
\begin{aligned}
\text{Area } P \;=\; & [\text{Total area of rectangle}] \\
& - [\text{Area of 2 pairs of equal triangles } T_1 \text{ and } T_2] \\
& - [\text{Area of 2 rectangles } R] \;.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\text{Area } P &= (a + b)(c + d) - 2 \cdot \frac{1}{2} ac - 2 \cdot \frac{1}{2} bd - 2bc \\
&= ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \mathbf{M}
\end{aligned}
$$

In view of the equation $\mathrm{d}x\,\mathrm{d}y = S\,\mathrm{d}u\,\mathrm{d}v$ one may understand this result such that the unit square of area $\mathrm{d}u\,\mathrm{d}v$ gets multiplied by a factor of $S = \det \mathbf{M}$. The same argument shows that a small rectangle of sides du and dv with area $du\,dv$ also gets multiplied by $S = \det \mathbf{M}$. Therefore, for a linear change of variables a small rectangular area $\mathrm{d}u\,\mathrm{d}v$ in the (u, v) plane is transformed into the parallelogram area $dx\,dy = \det \mathbf{M}\,du\,dv$ in the (x, y) plane.

Now let us consider a *nonlinear change* of coordinates. We take the transformation to have the form

$$x = x(u, v), \qquad y = y(u, v)\,,$$

where according to the total differential the increments in x and y are given by

$$\begin{aligned} dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \\ dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

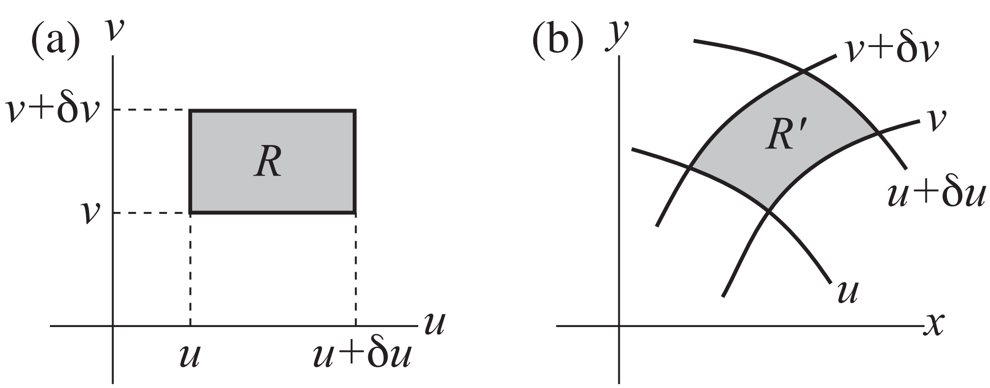
The **Jacobian matrix** is defined to be

$$\mathbf{M}(u, v) = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$

and the **Jacobian determinant**, or **Jacobian**,

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \mathbf{M}(u, v)\,.$$

This suggests that for a nonlinear change of variables we also have that a rectangular area $du\,dv$ in the (u, v) plane) is transformed into the (deformed) ‘parallelogram’ area $\det \mathbf{M} du\,dv$ in the (x, y) plane.



(with $du = \delta u$ $dv = \delta v$)

Therefore, the required formula for double integrals under a change of variables is:

$$\int\!\!\int_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int\!\!\int_{R'} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \mathrm{d}u\,\mathrm{d}v$$

where

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = |\det \mathbf{M}|$$

can be thought of as the scaling factor S.

Note that $|\cdot|$ denotes the absolute value of the determinant of the matrix, i.e., the modulus as in the one variable case. This may not be confused with the case of a matrix, where vertical lines on either side denote the determinant. For example, if we let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and

$$|\det \mathbf{A}| = |ad - bc| \,.$$

Example:
Evaluate the integral

$$I = \int\int_R (x^2 + y^2)\, \mathrm{d}x\, \mathrm{d}y$$

where R is a circle $x^2 + y^2 \leq a^2$, by changing to polar coordinates.
In polar coordinates we have

$$x = r\cos\theta, \qquad y = r\sin\theta\,.$$

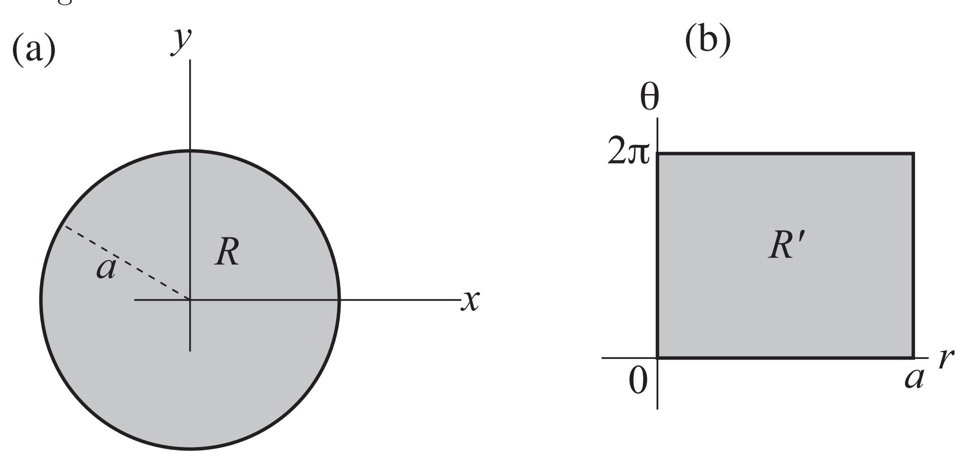
Therefore, taking $u = r$ and $v = \theta$, we can write the Jacobian matrix as

$$\mathbf{M} = \begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and the Jacobian determinant is

$$\det \mathbf{M} = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

where here and in the following we assume $r \geq 0$, so we do not need to take the absolute value. The original area R and the transformed area R' are shown below:



Note that the circle in the (x,y) plane transforms into a rectangle in the (r,θ) plane. Here R is the region given by $x^2 + y^2 \leq a^2$ and R' is the region given by $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$.

Therefore

$$I = \iint_R (x^2 + y^2)\, \mathrm{d}x\, \mathrm{d}y = \iint_{R'} (r^2)\,(r)\, \mathrm{d}r\, \mathrm{d}\theta$$

where the r^2 on the right-hand integral comes from the transformed $x^2 + y^2$ and the $r\, \mathrm{d}r\, \mathrm{d}\theta$ is from the transformed $\mathrm{d}x\, \mathrm{d}y$ with r coming from the Jacobian determinant $\det \mathbf{M}$. Hence

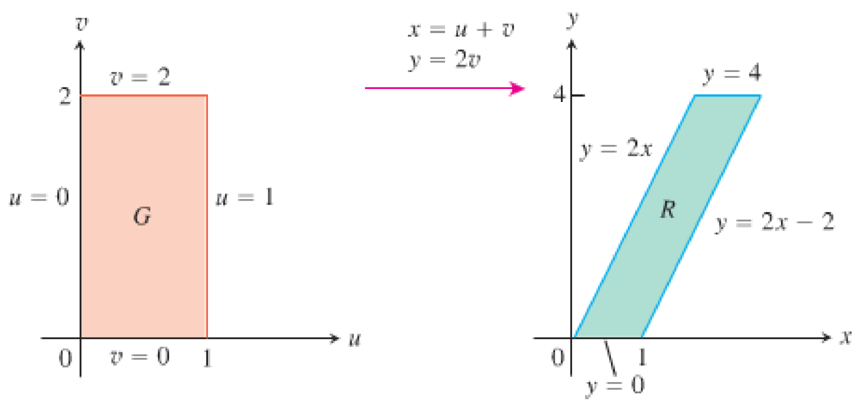
$$I = \int_{r=0}^{r=a} \int_{\theta=0}^{\theta=2\pi} r^3\, \mathrm{d}\theta\, \mathrm{d}r = \left(\int_{r=0}^{r=a} r^3\, \mathrm{d}r \right) \left(\int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \right) = \frac{\pi a^4}{2},$$

where we note that the integral is separable.

Example:
Evaluate the double integral

$$\int_0^4 \int_{x=y/2}^{x=y/2+1} \frac{2x - y}{2}\, \mathrm{d}x\, \mathrm{d}y$$

by applying the transformation $u = (2x - y)/2$, $v = y/2$ and integrating over an appropriate region of the u-v plane.
The region R in the x-y-plane looks as follows:



The corresponding region G in the u-v plane can be obtained by first writing x and y in terms of u and v as $x = u + v$ and $y = 2v$.
The boundaries of G are then found by substituting these equations for the boundaries of R:

xy*-equations for the boundary of *R	**Corresponding *uv*-equations for the boundary of *G***	**Simplified *uv*-equations**
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

The Jacobian of the transformation is

$$\begin{aligned} \det \mathbf{M}(u, v) &= \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} \\ &= \begin{vmatrix} \partial(u+v)/\partial u & \partial(u+v)/\partial v \\ \partial(2v)/\partial u & \partial(2v)/\partial v \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2\,. \end{aligned}$$

and we get

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2} \mathrm{d}x\,\mathrm{d}y = \int_{v=0}^{v=2}\int_{u=0}^{u=1} u\,|\det \mathbf{M}(u,v)|\;\mathrm{d}u\,\mathrm{d}v = \int_{v=0}^{v=2}\int_{u=0}^{u=1} u\cdot 2\,\mathrm{d}u\,\mathrm{d}v = 2$$

Note that for invertible transformations

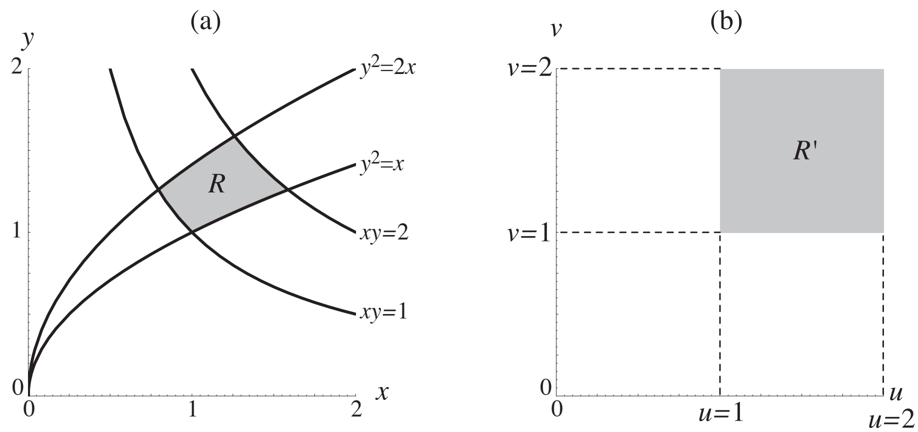
$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}, \tag{1}$$

as you have seen in Calculus 1 for a function of one variable. This can be useful in solving some problems.

Example:
Evaluate the integral

$$I = \int\int_R 1\cdot \mathrm{d}x\,\mathrm{d}y$$

(i.e. the area of the region R) where R is enclosed by $y^2 = x$, $y^2 = 2x$, $xy = 1$ and $xy = 2$.



To solve the integral consider the change of variables defined by

$$u = y^2/x, \qquad v = xy\,.$$

Then we can write the four bounding curves as

$$y^2 = x \Leftrightarrow u = 1, \quad y^2 = 2x \Leftrightarrow u = 2, \quad xy = 1 \Leftrightarrow v = 1, \quad xy = 2 \Leftrightarrow v = 2\,.$$

So the region becomes a square (the region R' in part (b) of the above figure).
Now, for the Jacobian determinant it is easier to use Eq. (1) above. So, to calculate $\partial(x,y)/\partial(u,v)$ we first calculate $\partial(u,v)/\partial(x,y)$ and then take the inverse. Using $u = y^2/x$ and $v = xy$ we have

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} -y^2/x^2 & 2y/x \\ y & x \end{vmatrix} = -3\frac{y^2}{x} = -3u\,.$$

Therefore, using Eq. (1),

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = -\frac{1}{3u}\,.$$

Hence

$$\begin{aligned}
I &= \iint_R 1\cdot \mathrm{d}x\,\mathrm{d}y = \iint_{R'} 1\cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v \\
&= \iint_{R'} \left|-\frac{1}{3u}\right| \mathrm{d}u\,\mathrm{d}v = \frac{1}{3}\int_{u=1}^{u=2}\int_{v=1}^{v=2} \frac{1}{u}\,\mathrm{d}v\,\mathrm{d}u \\
&= \frac{1}{3}\int_{u=1}^{u=2} \left[\frac{v}{u}\right]_{v=1}^{v=2} \mathrm{d}u \\
&= \frac{1}{3}\int_{u=1}^{u=2} \frac{1}{u}\,\mathrm{d}u = \frac{1}{3}\left[\ln u\right]_{u=1}^{u=2} = \frac{\ln 2}{3}
\end{aligned}$$

Reading assignment: Work yourself through the following example.

Example:
Evaluate the integral

$$\int_{-\infty}^{\infty} e^{-x^2/2}\mathrm{d}x\,.$$

If we call this integral I, we can write

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\mathrm{d}y\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\mathrm{d}x\,\mathrm{d}y\,.$$

Now transform to polar coordinates with the limits $0 \le r < \infty$ and $-\pi \le \theta \le \pi$. This gives

$$\begin{aligned}
I^2 &= \int_{-\pi}^{\pi}\int_{0}^{\infty} e^{-r^2/2}\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \mathrm{d}r\,\mathrm{d}\theta = \int_{-\pi}^{\pi}\int_{0}^{\infty} r\,e^{-r^2/2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{-\pi}^{\pi}\left[-e^{-r^2/2}\right]_0^{\infty} \mathrm{d}\theta = \int_{-\pi}^{\pi} \left((0)-(-1)\right)\,\mathrm{d}\theta = \int_{-\pi}^{\pi} \mathrm{d}\theta = 2\pi\,.
\end{aligned}$$

Hence $I = \sqrt{2\pi}$.
Note that the probability density function for a normal (or Gaussian) distribution is

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$$

for mean μ and standard deviation σ. If we write $t = (x-\mu)/\sigma$ (i.e. express the displacement from the mean in terms of the standard deviation) then the total probability is

$$\begin{aligned}
P &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\mathrm{d}x = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\sigma\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\,\mathrm{d}t = 1\,. \quad \text{(by our previous result)}
\end{aligned}$$