MTH4100 Calculus I

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What is Calculus?

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Calculus provides powerful techniques for solving problems which have widespread applications throughout science, economics, and engineering. It has been formalised and extended into the important branch of mathematics known as *analysis*.

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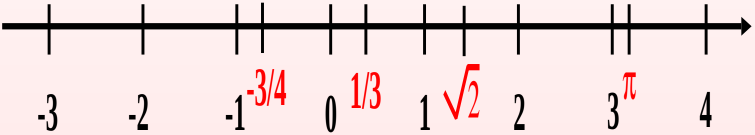
The real numbers can be represented as points on the *real line*.



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(M4) For all $a \in \mathbb{R}$ with $a \neq 0$, there is an element $a^{-1} \in \mathbb{R}$ such that $a\,a^{-1} = 1$. *inverse*

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Properties A0-A5, M0-M5, and D define an algebraic structure called a *field*.

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(O5) if $a \leq b$ and $0 \leq c$ then $a\,c \leq b\,c$ *order under multiplication*

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Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
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We can *prove* that these rules are valid by using properties O1-O5.

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If a set of real numbers S has an *upper bound* i.e. there exists a number $c \in \mathbb{R}$ such that $x \leq c$ for all $x \in S$, then S has a *least upper bound* i.e. there exists an upper bound c_0 for S such that $c \geq c_0$ for all upper bounds c of S.

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The completeness property tells us that an interval which is bounded above has a least upper bound. Similarly an interval which is bounded below has a greatest lower bound. We refer to these values as *end-points* of the interval.

Examples

- $I = \{x \in \mathbb{R} : 3 < x \leq 6\}$ defines a bounded interval. Geometrically, it corresponds to a *line segment* on the real line. It has two end-points 3 and 6. We can describe it using the notation $I = (3, 6]$, where the round bracket on the left tells us that $3 \notin I$ and the square bracket on the right tells us that $6 \in I$.

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- $I = \{x \in \mathbb{R} : x > -2\}$ defines an unbounded interval. Geometrically, it corresponds to a *ray* i.e. a line which extends to infinity in one direction. It has one end-point -2. We can describe it using the notation $I = (-2, \infty)$.

We can distinguish between intervals which are bounded or unbounded. We can also distinguish between intervals by considering whether or not they contain their end points: intervals which contain all their end-points are *closed*; intervals which contain none of their end-points are *open*; intervals which have two end points and contain exactly one of them are *half-open* (or *half-closed*).

Types of intervals

TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
	$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

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- $\frac{6}{x-1} \geq 5$. Since $\frac{6}{x-1} > 0$ we have $x - 1 > 0$ and hence $x > 1$. We can now use property (O5) to deduce that $6 \geq 5x - 5$ and hence $\frac{11}{5} \geq x$. Combining these two inequalities we see that the set of solutions is the interval $(1, \frac{11}{5}]$.

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- $x^2 - 2x - 1 > 2$. Then $x^2 - 2x - 3 > 0$ so $(x + 1)(x - 3) > 0$. Hence either $(x + 1)$ and $(x - 3)$ are both positive i.e. $x > 3$, or $(x + 1)$ and $(x - 3)$ are both negative i.e. $x < -1$. Thus the set of solutions is union of the two disjoint intervals $(-\infty, -1)$ and $(3, \infty)$.

Absolute Value

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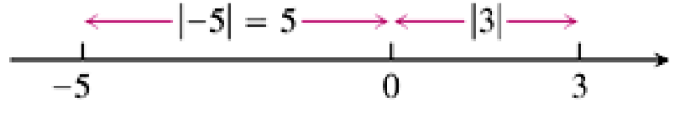
Geometrically, $|x|$ is the distance on the real line between x and 0.

example:



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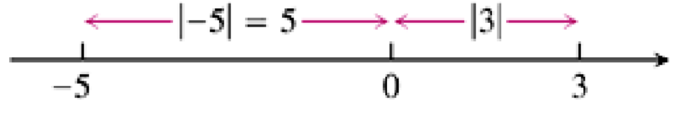

Similarly, for any $x, y \in \mathbb{R}$, $|x - y|$ is the distance between x and y.

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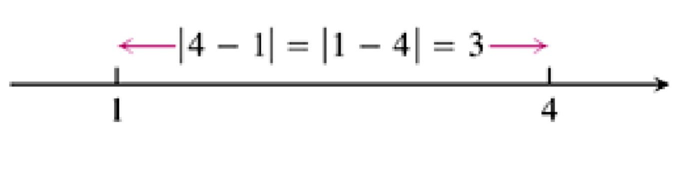

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Proof of (1). By definition, the symbol $\sqrt{a^2}$ is always taken to be the non-negative square root of a^2. So $\sqrt{a^2} = a$ if $a \geq 0$ and $\sqrt{a^2} = -a$ if $a < 0$. Hence $|a| = \sqrt{a^2}$. □

We can use (1) to prove (2)-(5).

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Lemma (Absolute values and Intervals) Suppose a is a positive real number. Then:

1. $|x| = a \Leftrightarrow x = \pm a$;
2. $|x| < a \Leftrightarrow -a < x < a \Leftrightarrow x \in (-a, a)$;
3. $|x| > a \Leftrightarrow x < -a$ or $x > a \Leftrightarrow x \in (-\infty, -a) \cup (a, \infty)$;
4. $|x| \leq a \Leftrightarrow -a \leq x \leq a \Leftrightarrow x \in [-a, a]$;
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Proof of (4). This follows because the distance from x to 0 is less than or equal to a if and only if x lies between a and $-a$. □

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Reading Assignment:

Thomas' Calculus, Appendix 3: Lines, Circles, and Parabolas