CW 7

1.) $f(x) = \frac{12}{3+x^2}$

i) the domain of f is $\mathbb{R}$

symmetries: $f(-x) = \frac{12}{3+(-x)^2} = \frac{12}{3+x^2}$

$= f(x)$

$\Rightarrow f$ is even

ii) $f'(x) = -\frac{12}{(3+x^2)^2} \cdot 2x = -\frac{24x}{(3+x^2)^2}$

$$f''(x) = -24 \frac{(3+x^2)^2 - x \cdot 2(3+x^2) \cdot 2x}{(3+x^2)^4}$$

$$= -24 \frac{3+x^2 - 4x^2}{(3+x^2)^3}$$

$$= 72 \frac{x^2 - 1}{(3+x^2)^3}$$

iii) critical points: $f'(x) = 0$

$\Rightarrow \quad x = 0$

$f''(0) = -\frac{72}{27} = -\frac{8}{3} < 0$

$\Rightarrow$ f has a maximum at $x = 0$,

$f(0) = 4$

iv) $f'(x) < 0$ for $x > 0$: decreasing

$f'(x) > 0$ for $x < 0$: increasing

v) points of inflection: $f''(x) = 0$

$\Leftrightarrow \quad x^2 - 1 = 0$

$\Rightarrow$ possible points of inflection at $x = \pm 1$

$f(1) = 3 \quad , \quad f(-1) = 3$

$f''(x) > 0$ for $x > 1$ or $x < -1$: concave up

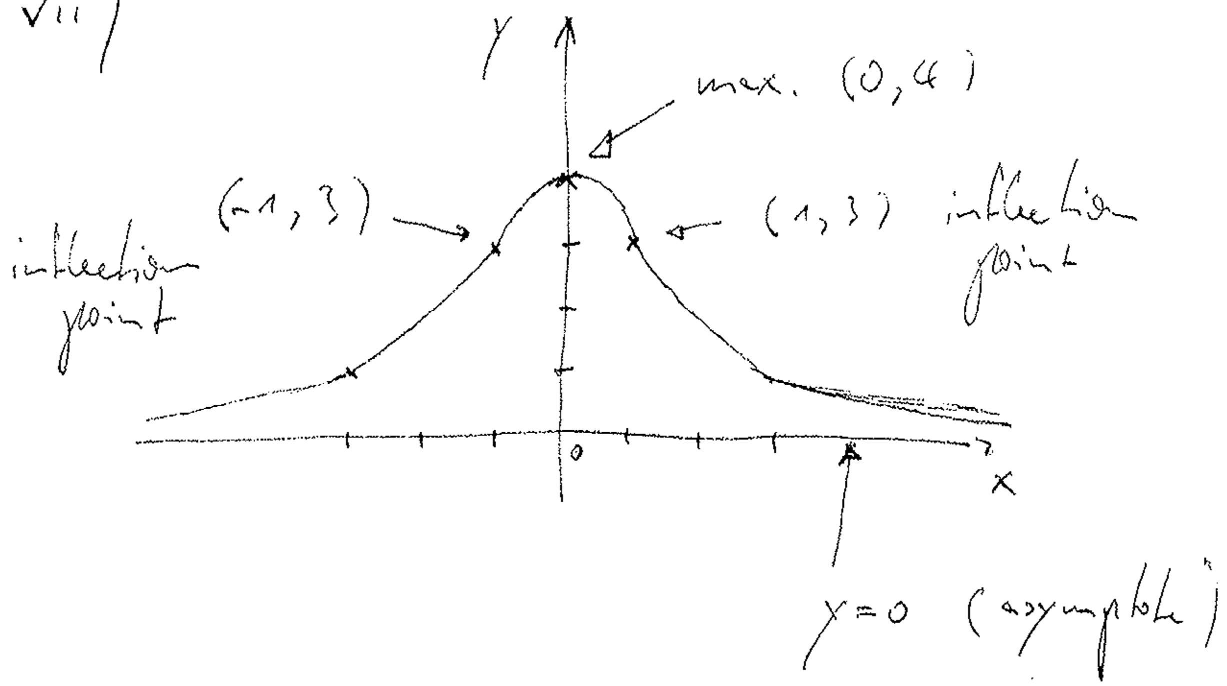
$f''(x) < 0$ for $-1 < x < 1$: concave down

v) sign change of $f''(x)$ at $x=1$ and $x=-1$: inflection points confirmed

vi) no vertical asymptotes

$\lim_{x \to \pm\infty} f(x) = 0 \Rightarrow y = 0$ is horizontal asymptote

vii)



2.) $f(x) = -5x^2 + \frac{1}{4} x^4$

i) the domain of f is $\mathbb{R}$

symmetries : $f(-x) = -5(-x)^2 + \frac{1}{4}(-x^4)$

$= -5x^2 + \frac{1}{4}x^4$

$= f(x)$

$\Rightarrow f$ is even

ii) $f'(x) = -10x + x^3$

$f''(x) = -10 + 3x^2$

iii) critical points : $f'(x) = 0$

$\Rightarrow x(-10 + x^2) = 0$

$\Rightarrow x = 0$, or $x = \pm\sqrt{10}$

$f''(0) = -10 < 0$: $x = 0$ local max.

$f''(\sqrt{10}) = f''(-\sqrt{10}) = 20 > 0$:

$x = \pm\sqrt{10}$ local minima

$f(\pm\sqrt{10}) = -25$

iv) $f'(x) < 0$ for $x < -\sqrt{10}$

and $0 < x < \sqrt{10}$

$\Rightarrow$ f is decreasing

$f'(x) > 0$ for $-\sqrt{10} < x < 0$

and $\sqrt{10} < x$

$\Rightarrow$ f is increasing

v) $f''(x) = 0 \iff 3x^2 = 10$

$\iff x = \pm\sqrt{\frac{10}{3}}$

for $x < -\sqrt{\frac{10}{3}}$, $f''(x) > 0$

$\Rightarrow$ f is concave up

for $-\sqrt{\frac{10}{3}} < x < \sqrt{\frac{10}{3}}$, $f''(x) < 0$

$\Rightarrow$ f is concave down

for $\sqrt{\frac{10}{3}} < x$, $f''(x) > 0$

$\Rightarrow$ f is concave up

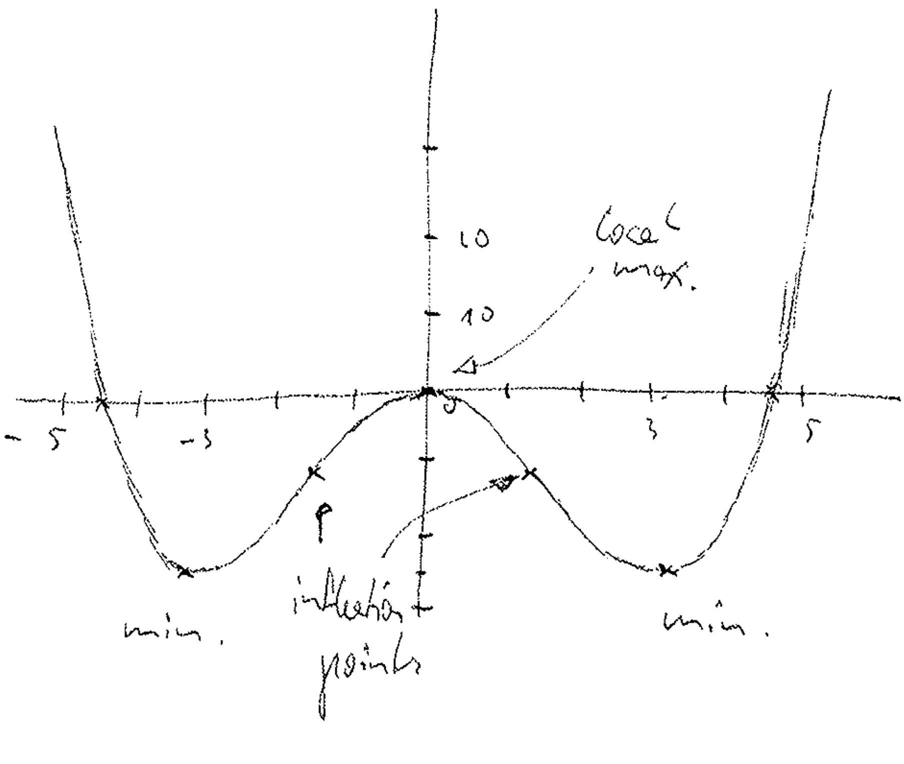
v) hence $x = \pm \sqrt{\frac{10}{3}}$ are inflection points, $f\left(\sqrt{\frac{10}{3}}\right) = f\left(\sqrt{\frac{10}{3}}\right)$
$= -\frac{125}{9}$

vi) no vertical asymptotes

$\lim_{x \to \pm\infty} f(x) = +\infty \implies$ no horizontal asymptotes

and no oblique asymptotes

vii) $f(x) = 0$: $x^2\left(-5 + \frac{1}{4}x^2\right) = 0$
(intercepts) $x = 0$, or $x = \pm\sqrt{20}$



3) Conditions; $x+y=20$,

$x \geq 0$, $y \geq 0$.

We want to maximise $g(x) = x + \sqrt{y}$ subject to these conditions.

Hence we want to maximise $g(x) = x + \sqrt{20-x}$

for $x \in [0, 20]$.

<u>Find critical points</u>:

$$g'(x) = 1 - \frac{1}{2\sqrt{20-x}} = \frac{2\sqrt{20-x} - 1}{2\sqrt{20-x}}$$

Hence $g'(x) = 0$ when $2\sqrt{20-x} = 1$ ie $x = 79/4$

and $g'(x)$ does not exist when $x = 20$.

So we have two critical points $x = 79/4, 20$.

<u>Evaluate g at critical points and end-points</u>

$g(0) = \sqrt{20}$, $g(79/4) = 81/4$, $g(20) = 20$

So maximum value is $81/4$ and this occurs when $x = 79/4$ and $y = 1/4$.