1) Prove that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for all $a, b \in \mathbb{R}$ with $b \neq 0$.

<u>Proof</u> By Lemma 1.1, $|x| = \sqrt{x^2}$.

Hence

$$\left|\frac{a}{b}\right| = \sqrt{\left(\frac{a}{b}\right)^2} = \sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \frac{|a|}{|b|}.$$

$\blacksquare$

<u>Note</u> Lemma 1.3 tells us that $|ac| = |a||c|$. We could try to use this by putting $c = \frac{1}{b}$. This gives

$$\left|\frac{a}{b}\right| = |ac| = |a||c| = |a|\,|1/b|$$

BUT we still have to show that $\left|\frac{1}{b}\right| = \frac{1}{|b|}$.

2.) What $x \in \mathbb{R}$ satisfy $x^2 - 4x - 12 < 0$

a) $x^2 - 4x - 12 < 0$

$x^2 - 4x + 4 - 4 - 12 < 0$

$(x-2)^2 - 16 < 0$

$(x-2)^2 < 16$ take square root, definition of mod.

$|x-2| < 4$ use $|y| < a \iff -a < y < a$

$-4 < (x-2) < 4$

$-2 < x < 6$

$\Rightarrow$ solution : $x \in (-2, 6)$ ← open set

b) see e.g. Appendix 3, AP-15

plot graph $y = x^2 - 4x - 12$

(rough sketch sufficient)

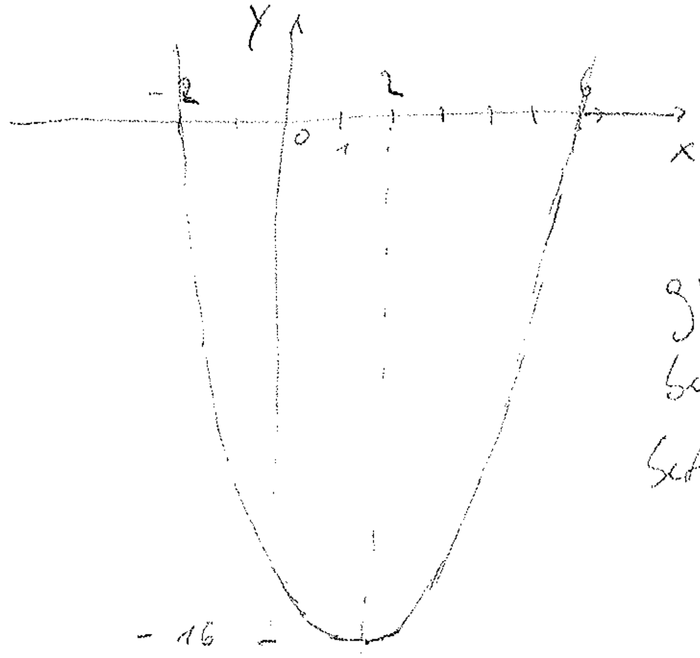
<u>either</u>:

x	0	1	2	3	4	5	...
y	-12	-15	-16	-15	-12	-7	

<u>or</u>: from above

$y = (x-2)^2 - 16$

parabola : shifted to right and down



graph ~~[illegible]~~ below x-axis between -2 and 6

Need intersection with x-axis : $y = 0$

$\Rightarrow \quad x^2 - 4x - 12 = 0$

Solve: $\quad x_{1/2} = \dfrac{+4 \pm \sqrt{16 + 48}}{2}$

$= \dfrac{4 \pm 8}{2}$

$= -2 \, , \, 6$

i.e. parabola intersects x-axis at $x_1 = -2$ and $x_2 = 6$.

$\Rightarrow \quad x \in (-2, 6)$ //

3.) Find the set of real numbers satisfying

$$|2x-1| + |4x+1| < 3$$

Recall: $$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Hence: distinguish four different cases.

i) $2x - 1 < 0$ and $4x+1 < 0$

$\Leftrightarrow$ $x < \frac{1}{2}$ and $x < -\frac{1}{4}$

i.e. $x \in (-\infty, -\frac{1}{4})$

In this case, get

$$\begin{aligned} &|2x-1| + |4x+1| \\ &= -(2x-1) - (4x+1) \\ &= -6x \end{aligned}$$

Hence $-6x < 3 \Leftrightarrow x > -\frac{1}{2}$

So, $x \in (-\infty, -\frac{1}{4}) \cap (-\frac{1}{2}, \infty)$

$$= (-\tfrac{1}{2}, -\tfrac{1}{4})$$

ii) $2x - 1 < 0$ and $4x + 1 \geq 0$

$\iff x < \frac{1}{2}$ and $x \geq -\frac{1}{4}$

i.e. $x \in [-\frac{1}{4}, \frac{1}{2})$

Get, $|2x-1| + |4x+1|$

$$= -(2x - 1) + 4x + 1$$
$$= 2x + 2$$

$2x + 2 < 3 \iff x < \frac{1}{2}$

So, $x \in [-\frac{1}{4}, \frac{1}{2}) \cap (-\infty, \frac{1}{2})$

$$= [-\frac{1}{4}, \frac{1}{2})$$

iii) $2x - 1 \geq 0$ and $4x + 1 < 0$

$\iff x \geq \frac{1}{2}$ and $x < -\frac{1}{4}$

Contradiction! Not possible for any $x \in \mathbb{R}$.

iv) $2x - 1 \geq 0$ and $4x + 1 \geq 0$

$\iff x \geq \frac{1}{2}$ and $x \geq -\frac{1}{4}$

i.e. $x \in [\frac{1}{2}, \infty)$

Get $|2x - 1| + |4x + 1|$

$= 2x - 1 + 4x + 1 = 6x$

Hence, $6x < 3 \iff x < \frac{1}{2}$

and $x \in [\frac{1}{2}, \infty) \cap (-\infty, \frac{1}{2}) = \emptyset$

"empty set"

Put finally everything together:

$x \in (-\frac{1}{2}, -\frac{1}{4}) \cup [-\frac{1}{4}, \frac{1}{2})$

$= (-\frac{1}{2}, \frac{1}{2})$

4.) Find all $x \in \mathbb{R}$ satisfying

$$\sqrt{1-x^2} \leq -x \qquad (*)$$

a) Note: need $1-x^2 \geq 0$ or square root on LHS undefined, i.e.

$$x^2 \leq 1 \iff |x| \leq 1$$

$$\iff -1 \leq x \leq 1$$

Also: LHS positive or zero, hence $x \leq 0$

Finally: square both sides of $(*)$, get

$$1-x^2 \leq x^2$$

$$\iff 2x^2 \geq 1 \iff x^2 \geq \frac{1}{2}$$

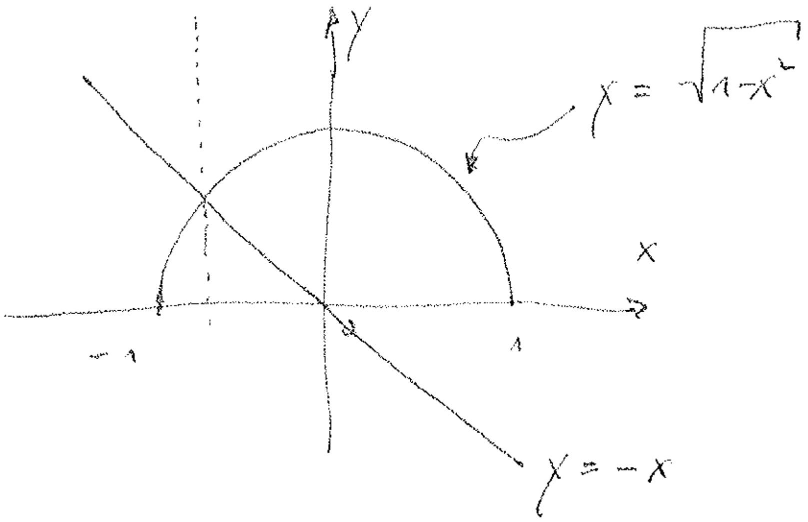
$$\iff |x| \geq \sqrt{\tfrac{1}{2}} \iff x \geq \sqrt{\tfrac{1}{2}} \text{ or } x \leq -\sqrt{\tfrac{1}{2}}$$

Using the above, i.e. $x \in [-1, 1]$,

get $$x \in \left[-1, -\frac{1}{\sqrt{2}}\right]$$

4.) b) $y = -x$: straight line, slope -1, through origin

$y = \sqrt{1-x^2}$: upper half of unit-circle



intersection of the two graphs at

$$\sqrt{1-x^2} = -x$$

$\Leftrightarrow \quad 1 - x^2 = x^2 \quad \Leftrightarrow \quad x^2 = \frac{1}{2}$

$\Leftrightarrow \quad x_{1/2} = \pm \frac{1}{\sqrt{2}}$

must choose $x = -\frac{1}{\sqrt{2}}$, see graph above

Find again, $x \in \left[-1, -\frac{1}{\sqrt{2}}\right]$. ✓