MTH 4101 Calculus II, Sample Exam

Model Solutions

(Please note: This exam is identical to the exam from 2012 apart from questions 1.(a), (b) and 2. They have been replaced due to a change of syllabus.)

1.(a) use l'Hôpital: [2] $\lim_{n\to\infty} \frac{n}{5^n} = \lim_{n\to\infty} \frac{n}{e^{n\ln 5}} = \lim_{n\to\infty} \frac{1}{\ln 5 \cdot 5^n}$ [3]

$= \underline{0}$ converges [2] $\boxed{\text{new}}$

(b) $\frac{dy}{dx} = e^{x-y}$: rewrite as $e^y dy = e^x dx$ [2]

Apply separation of variables technique:

$\int e^y dy = e^y + C_1$, $\int e^x dx = e^x + C_2$ [3] $\boxed{\text{new}}$

$\Rightarrow$ $\underline{\underline{e^y - e^x = C_2 - C_1 = C}}$ [2]

(c) The domain of f does not include the y-axis where $x=0$. But it does include the x-axis where $f(x,0)=0$ $(x\neq 0)$.

Hence, $\lim_{(x,0)\to(0,0)} f(x,y) = \lim_{(x,0)\to(0,0)} \frac{y}{x} = 0$ [3] $\boxed{\text{new but seen similar}}$

However, for a path along $y=x$ we have

$$\lim_{(x,x)\to(0,0)} f(x,y) = \lim_{(x,x)\to(0,0)} \frac{y}{x} = 1 \quad \boxed{4}$$

Therefore, the limit of $f(x,y)$ does not exist as $(x,y)\to(0,0)$.

~~[Give 2 marks if it has been stated correctly that the limit does not exist.]~~

(d) $f_x \overset{\boxed{1}}{=} -7\cos(yz)e^x$, $f_y \overset{\boxed{1}}{=} 7e^x z\sin(yz)$, $f_z \overset{\boxed{1}}{=} 7e^x y\sin(yz)$

7 $\quad f_x|_{(0,0,0)} = -7$; $f_y|_{(0,0,0)} = 0$; $f_z|_{(0,0,0)} = 0$

$\Rightarrow \underline{\nabla} f|_{(0,0,0)} = -7\underline{i}$ $\boxed{1}$

Seen very similar in coursework and lectures

$$|\underline{A}| = \sqrt{(4)^2+(3)^2+(-5)^2} = 5\sqrt{2} \quad \boxed{1}$$

$$\Rightarrow \underline{u} = \frac{\underline{A}}{|\underline{A}|} = \frac{4\underline{i}+3\underline{j}-5\underline{k}}{5\sqrt{2}} \quad \boxed{1}$$

$$D_{\underline{u}} f|_{(0,0,0)} = \underline{\nabla} f|_{(0,0,0)} \cdot \underline{u} = \frac{(-7)\cdot 4}{5\sqrt{2}} = -\frac{14\sqrt{2}}{5} \quad \boxed{1}$$

(e) $f_x \overset{\boxed{1}}{=} 2y-10x+4$; $f_y \overset{\boxed{1}}{=} 2x-4y+4$

similar to coursework & lecture

7 at extreme values we have $f_x = f_y = 0$ $\boxed{1}$ $\Rightarrow \begin{cases} -5x+y=-2 \\ x-2y=-2 \end{cases}$

Hence, $x = \frac{2}{3}$ and $y = \frac{4}{3}$. $\boxed{1}$

$f_{xx} = -10$, $f_{yy} = -4$, $f_{xy} = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 40-4 = 36 > 0$ $\boxed{1}$

$f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0 \Rightarrow$ local max. at $\left(\frac{2}{3},\frac{4}{3}\right)$ $\boxed{1}$

with $f\left(\frac{2}{3},\frac{4}{3}\right) = \frac{16}{9} - \frac{20}{9} - \frac{32}{9} + \frac{8}{3} + \frac{16}{3} - 4 = 0$ $\boxed{1}$

(h) $\int_0^1\int_y^1 x^2 e^{xy}\,dx\,dy \overset{[2]}{=} \int_0^1\int_0^x x^2 e^{xy}\,dy\,dx$ ||cp. cons sem./ lect.||

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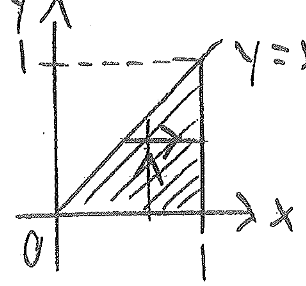
[3]



$$= \int_0^1 \left[x e^{xy}\right]_0^x dx = \int_0^1 \left(x e^{x^2} - x\right) dx \quad [1]$$

$$= \left[\tfrac{1}{2} e^{x^2} - \tfrac{1}{2}x^2\right]_0^1 = \underline{\underline{\tfrac{e}{2} - 1}} \quad [1]$$

(f) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2^n} - \frac{1}{4^n}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n} - \sum_{n=0}^{\infty}\frac{1}{4^n}$ ||almost identical to cw/ lect.||

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Each is a geometric series: $\displaystyle\sum_{n=0}^{\infty} ar^n \overset{[2]}{=} \frac{a}{1-r}$

$$\Rightarrow \sum_{n=0}^{\infty}\left(-\tfrac{1}{2}\right)^n = \frac{1}{1+\frac{1}{2}} \overset{[2]}{=} \frac{2}{3}\ ;\ \sum_{n=0}^{\infty}\left(\tfrac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3} \quad [2]$$

Hence, $\underline{\underline{\displaystyle\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2^n} - \frac{1}{4^n}\right)}} = \frac{2}{3} - \frac{4}{3} = \underline{\underline{-\frac{2}{3}}}$ [1]

(g) $f'(x) \overset{[2]}{=} -\dfrac{\sin x}{\cos x} = -\tan x\ ,\ f''(x) = -\dfrac{1}{\cos^2 x} = -\sec^2 x$ [2]

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$f(0) = \ln 1 = 0\ ;\ f'(0) = 0\ ;\ f''(0) = -1$ [1]

||seen similar||

$\Rightarrow \underline{\underline{P_1(x)}} = f(0) + x f'(0) = \underline{\underline{0}}$ [1]

$\underline{\underline{P_2(x)}} = P_1(x) + \frac{1}{2}x^2 f''(0) = \underline{\underline{-\frac{1}{2}x^2}}$ [1]

2. Let $\{a_n\}$ be a sequence of positive terms. If $a_n = f(n)$,
(11) where f is a continuous, positive, decreasing function of x for all $x \geq N \in \mathbb{N}$, then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ both converge or both diverge. $\boxed{3}$ $\| \boxed{\text{bookwork}} \|$

$f(x) = \frac{\ln(x+1)}{x+1}$, $x \geq 3$, is continuous, positive, and decreasing, hence fulfills the assumptions of this test.

Calculate $\int_3^a \frac{\ln(x+1)}{x+1}\,dx = \frac{\ln(x+1)^2}{2}\Big|_3^a$ $\boxed{3}$ $\| \boxed{\text{Seen similar}} \|$

But $\lim_{a \to \infty} \frac{1}{2}\ln(x+1)^2\Big|_3^a = \infty$. $\boxed{3}$

That is, the improper integral diverges, thus the given series also diverges. $\boxed{2}$

3. $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$
(11)

$$= (y+z)\cdot 1 + (x+z)\cdot 1 + (y+x)\cdot v$$

$$= u - \cancel{v} + uv + u + \cancel{v} + uv + (u - \cancel{v} + u + \cancel{v})v$$

$$= \underline{\underline{2u + 4uv}} \quad \boxed{4}$$

$\| \boxed{\text{Seen similar but "new"}} \|$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$= (y+z)\cdot 1 + (x+z)(-1) + (y+x)\cdot u$$

$$= \cancel{u}-v + \cancel{uv} - (\cancel{u}+v+\cancel{uv}) + (u-\cancel{v}+u+\cancel{v})u$$

$$= \underline{-2v + 2u^2} \quad \boxed{4}$$

Alternatively, $w = (u+v)(u-v) + (u-\cancel{v})uv + (u+\cancel{v})uv$

$$= u^2 - v^2 + 2u^2 v$$

Hence, $\frac{\partial w}{\partial u} = \underline{2u + 4uv}$ and $\frac{\partial w}{\partial v} = \underline{-2v + 2u^2}$ $\boxed{3}$

4. $f(x,y) = 5 - x^2 - y^2$, $x = 2 + 2y \Rightarrow f(y) = 5 - (2+2y)^2 - y^2 = 1 - 8y - 5y^2$ $\boxed{1}$

(11) $f_y = -8 - 10y = 0 \Rightarrow y = -\frac{4}{5}$ $\boxed{1}$

$x = 2 + 2\cdot\left(-\frac{4}{5}\right) = \frac{2}{5}$ $\boxed{1}$; $f = 1 + 8\cdot\frac{4}{5} - 5\left(\frac{4}{5}\right)^2 = \frac{21}{5}$ $\boxed{1}$

$f_{yy} = -10 < 0$: f has a maximum. $\boxed{1}$ ||"new"||

Alternatively, $f_x = -2x$; $f_y = -2y$; $g_x = 1$; $g_y = -2$ $\boxed{2}$

$f_x = \lambda g_x \Rightarrow -2x = \lambda$; $f_y = \lambda g_y \Rightarrow -2y = -2\lambda$ $\boxed{1}$

in g: $x - 2y = -\frac{\lambda}{2} - 2\lambda = 2 \Rightarrow \lambda = -\frac{4}{5}$ $\boxed{1}$

Hence, $x = -\frac{\lambda}{2} = \frac{2}{5}$ and $y = \lambda = -\frac{4}{5}$ $\boxed{1}$

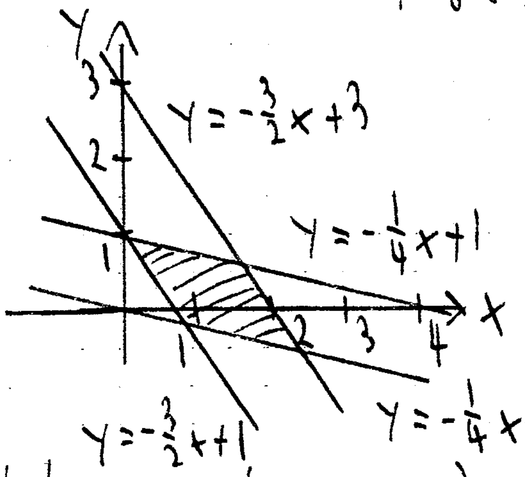
$f(x,y) = 5 - \left(\frac{2}{5}\right)^2 - \left(-\frac{4}{5}\right)^2 = 5 - \frac{4}{25} - \frac{16}{25} = \frac{21}{5}$ $\boxed{1}$

||similar to lect. & coursew.||

5. $x = v - 4y \Rightarrow u = 3(v-4y) + 2y = 3v - 10y$

(11) $\Rightarrow \underline{\underline{y}} = \underline{\underline{\dfrac{3v-u}{10}}}$; $\underline{\underline{x}} = v - \cancel{4}_{2}\left(\dfrac{3v-u}{\cancel{10}_{5}}\right) = \underline{\underline{\dfrac{2u-v}{5}}}$ $\boxed{2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{vmatrix} = \frac{6}{50} - \frac{1}{50} = \underline{\underline{\frac{1}{10}}} \ \boxed{2}$$



(diagram not necessary)

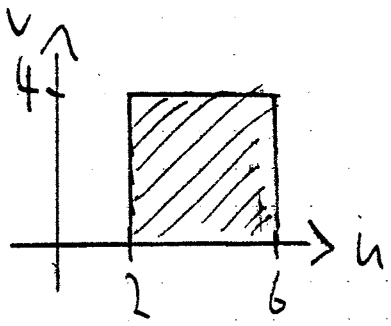
$y = -\frac{3}{2}x + 1 \Rightarrow u = 2$

$y = -\frac{3}{2}x + 3 \Rightarrow u = 6$ $\boxed{2}$

$y = -\frac{1}{4}x \Rightarrow v = 0$

$y = -\frac{1}{4}x + 1 \Rightarrow v = 4$

$\Rightarrow$



$$\iint_R (3x+2y)(x+4y)\,dx\,dy$$

$$= \int_{v=0}^{v=4} \int_{u=2}^{u=6} u \cdot v \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \ \boxed{2} = \frac{1}{10} \int_2^6 u\,du \int_0^4 v\,dv \ \boxed{1}$$

$$= \frac{1}{10}\left[\frac{u^2}{2}\right]_2^6 \cdot \left[\frac{v^2}{2}\right]_0^4 = \frac{1}{10}(18-2)(8-0) = \frac{\cancel{128}^{64}}{\cancel{10}_{5}} = \underline{\underline{\frac{64}{5}}} \ \boxed{2}$$

|| similar to lect. & coursew. ||