CHAP. 2. CURVES, LENGTHS, SURFACES & AREAS.

Recall the 3 ways to specify a curve in 2D x, y:

i) $y = f(x)$

ii) $g(x, y) = c$

iii) $x = f(t),\ y = g(t)$

t is PARAMETER.

t has defined DOMAIN
(eg $-\infty < t < \infty$, or $0 \leq t \leq 1$)

Curve is set of all points

$$(x, y) = (f(t), g(t))$$

for t in given domain.

Parametric curves can express complex curves which are very messy in the other forms.

Given a curve C, and a point P, it <u>may</u> be hard to find if given point P is on the curve!

eg given $P = (x_0, y_0)$,

solve for $f(t_0) = x_0$ <u>or</u> $g(t_0) = y_0$,

then test the other equation.

Parametric curves: Examples:

<u>Straight line:</u>

$$x = x_0 + at$$
$$y = y_0 + bt$$

Line through (x_0, y_0) with slope $\frac{b}{a}$.

Rearrange to $y = y_0 + (b/a)(x - x_0)$

Circle:

$$x^2 + y^2 = a^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

$x = a \overset{\cos}{\cancel{\sin}}\, t$, $y = a \sin t$

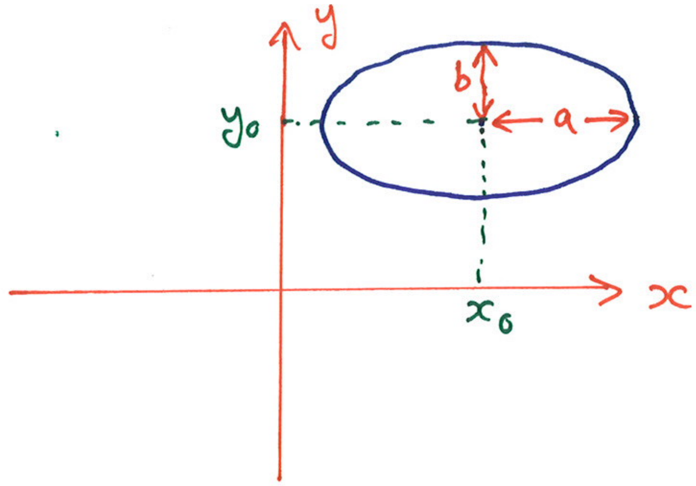
obviously satisfies this.

Ellipse : $x = a\cos t$, $y = b\sin t$

(centroid $(0,0)$)

Ellipse with centroid (x_0, y_0) :

$$x = x_0 + a\cos t$$
$$y = y_0 + b\sin t .$$



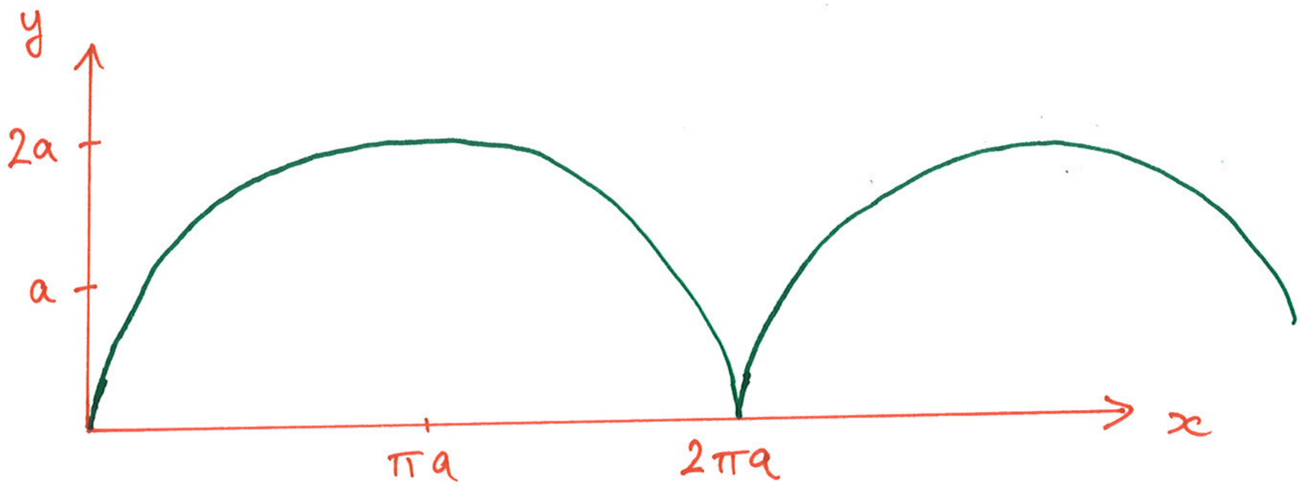
Note domain of t :
if $-\infty < t < \infty$, curve 'wraps round' ellipse infinitely many times.

To go round ellipse <u>exactly once</u>,
use $0 \leq t \leq 2\pi$.

The Cycloid :

$$x = a(t - \sin t)$$

$$y = a(1 - \cos t)$$



This is the curve traced by a point on a 'circular 'wheel' rolling along the x-axis.

[There are generalisations of circles rolling inside / outside another circle, called epicycloid and hypocycloid.].

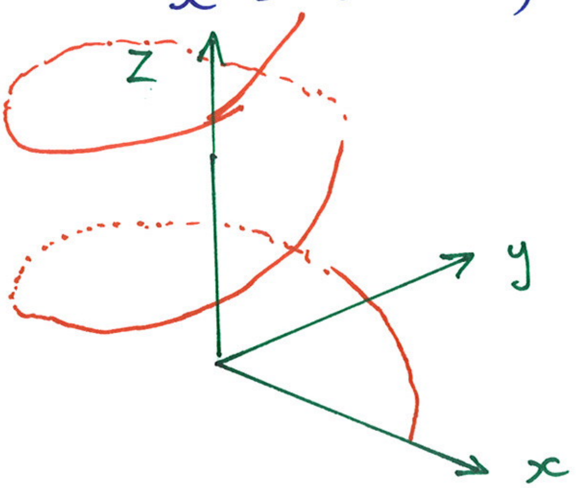
2.1.4 Parametric Curves in 3 dimensions.

It is simple to extend to 3-D by adding a 3rd function $z = h(t)$, so

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

Example: Helix.

$$x = a \cos t, \quad y = a \sin t, \quad z = bt.$$



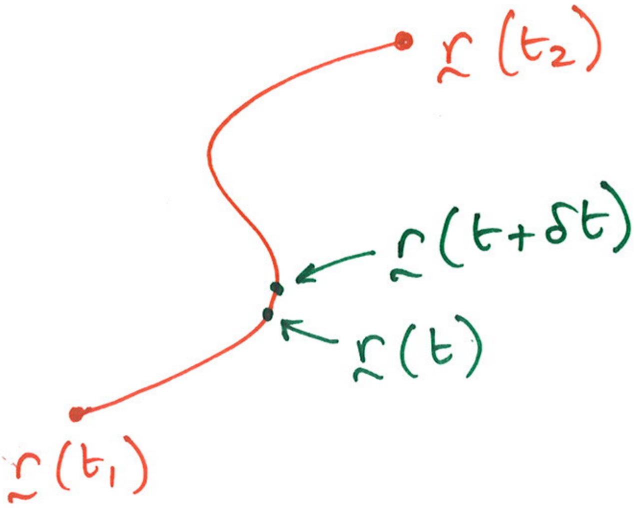
Projects onto a circle in x/y plane, so every point on curve is distance a from the z-axis, but its z-coordinate increases by $2\pi b$ per revolution.

2.2 ARC-LENGTH OF A CURVE.

Suppose we have a parametric curve given by

$$\underset{\sim}{r} = \underset{\sim}{r}(t) = \big(f(t),\, g(t),\, h(t)\big).$$

What is the length of the curve between points $\underset{\sim}{r}(t_1)$ and $\underset{\sim}{r}(t_2)$?



Consider a point $\underset{\sim}{r}(t)$ and a neighbouring point $\underset{\sim}{r}(t+\delta t)$:

vector distance between them is

$$\underset{\sim}{r}(t+\delta t) - \underset{\sim}{r}(t) \approx \frac{d\underset{\sim}{r}}{dt}\,\delta t$$

Let ds be the length of the above,

$$ds = \left|\underset{\sim}{r}(t+\delta t) - \underset{\sim}{r}(t)\right| = \left|\frac{d\underset{\sim}{r}}{dt}\right|\delta t.$$

So $$\frac{ds}{dt} = \left|\frac{d\underset{\sim}{r}}{dt}\right|$$

$$= \left|\left(\frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt}\right)\right|$$

$$= \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}$$

Get arc-length L by summing up all these infinitesimal line-segments,

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt$$

(in 2 dimensions, just set $z = h(t) = 0$)

Note :

you may be given a parametric curve with end points given as (x_1, y_1, z_1) and (x_2, y_2, z_2). In this case you need to solve for t_1 and t_2 giving those endpoints – solve whichever of the 3 eqns. is simplest, then check other two.

Example 2.1 :

The curve C is given by

$$x = t, \quad y = t^2, \quad z = \frac{2}{3}t^3 .$$

Evaluate the arc-length of C between points $(0,0,0)$ and $(2, 4, \frac{16}{3})$.

Since $x = t$, clearly $t_1 = 0$ and $t_2 = 2$.

Then $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2t$, $\frac{dz}{dt} = 2t^2$

$$L = \int_0^2 \sqrt{1^2 + (2t)^2 + (2t^2)^2} \, dt$$

$$= \int_0^2 \sqrt{1 + 4t^2 + 4t^4} \, dt$$

$$= \int_0^2 1 + 2t^2 \, dt$$

$$= \left[t + \frac{2}{3}t^3 \right]_0^2 = \frac{22}{3} .$$

2.2.1 TANGENT VECTOR TO A CURVE.

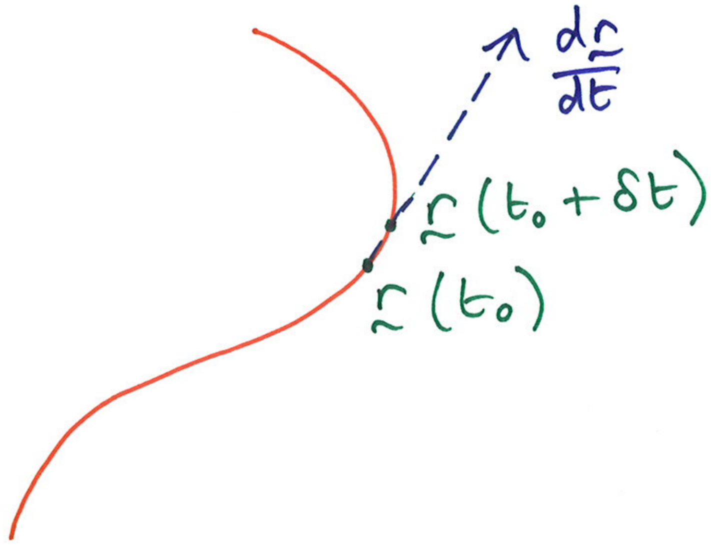
Given a parametric curve

$$\underset{\sim}{r} = \underset{\sim}{r}(t) = (f(t), g(t), h(t))$$

we can differentiate w.r.to t, giving

$$\frac{d\underset{\sim}{r}}{dt} \equiv \lim_{\delta t \to 0} \frac{\underset{\sim}{r}(t+\delta t) - \underset{\sim}{r}(t)}{\delta t}$$

$$= \left(\frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt}\right).$$



So tangent line to our curve at $\underset{\sim}{r}(t_0)$ goes through $\underset{\sim}{r}_0 \equiv r(t_0)$, and is parallel to $\left.\frac{d\underset{\sim}{r}}{dt}\right|_{t_0}$,

eqn. of line is $\underset{\sim}{r} = \underset{\sim}{r}_0 + u \left.\frac{d\underset{\sim}{r}}{dt}\right|_{t_0}$.

WARNING :

$$\frac{d\underset{\sim}{r}}{dt} = \left(\frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt}\right)$$

$$= \frac{df}{dt}\underset{\sim}{i} + \frac{dg}{dt}\underset{\sim}{j} + \frac{dh}{dt}\underset{\sim}{k}$$

looks a bit like

$$\underset{\sim}{\nabla} f = \frac{\partial f}{\partial x}\underset{\sim}{i} + \frac{\partial f}{\partial y}\underset{\sim}{j} + \frac{\partial f}{\partial z}\underset{\sim}{k}.$$

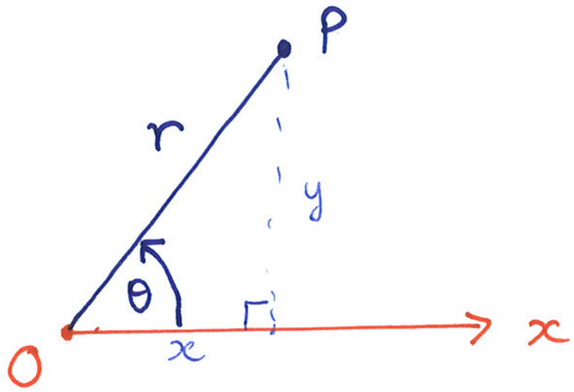
BUT

$\frac{d\underset{\sim}{r}}{dt}(t)$ has 3 fns (for x, y, z) of t, all differentiated w.r.t t.

$\underset{\sim}{\nabla} f$ has 1 fn $f(x,y,z)$ and partial derivatives.

2.3 CURVES IN POLAR COORDS.

Sometimes it is convenient to work in 2D plane polar coordinates: radius r and angle θ.



r is distance from origin

θ is angle <u>anticlockwise</u> from $+x$ axis.

Easy to see
$$x = r\cos\theta$$
$$y = r\sin\theta$$

Other way:
$$r = \pm\sqrt{x^2+y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right) + n\pi$$

Note $r,\theta \to x,y$ is unique

$x,y \to r,\theta$ is NOT unique

e.g (r,θ) and $(r, \theta+2n\pi)$

and $(-r, \theta+(2n+1)\pi)$

are the same point.

We can now define curves in polar coords as

$$r = f(\theta)$$

eg $\quad r = a + b\theta \quad$ – Archimedes spiral

$\quad\quad r = a e^{k\theta} \quad$ – logarithmic spiral.

2.3.1 CONICS IN POLAR COORDS

The polar equation

$$r = \frac{l}{1 + e\cos\theta}$$

Not 2.718!

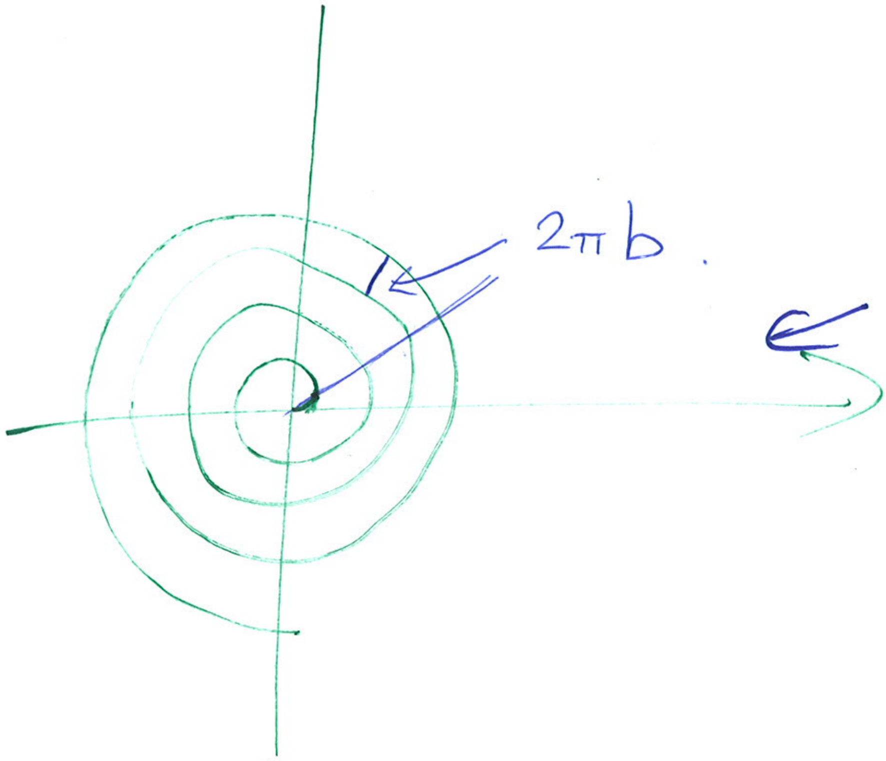
can give all the conic sections depending on e:

$e = 0$: circle

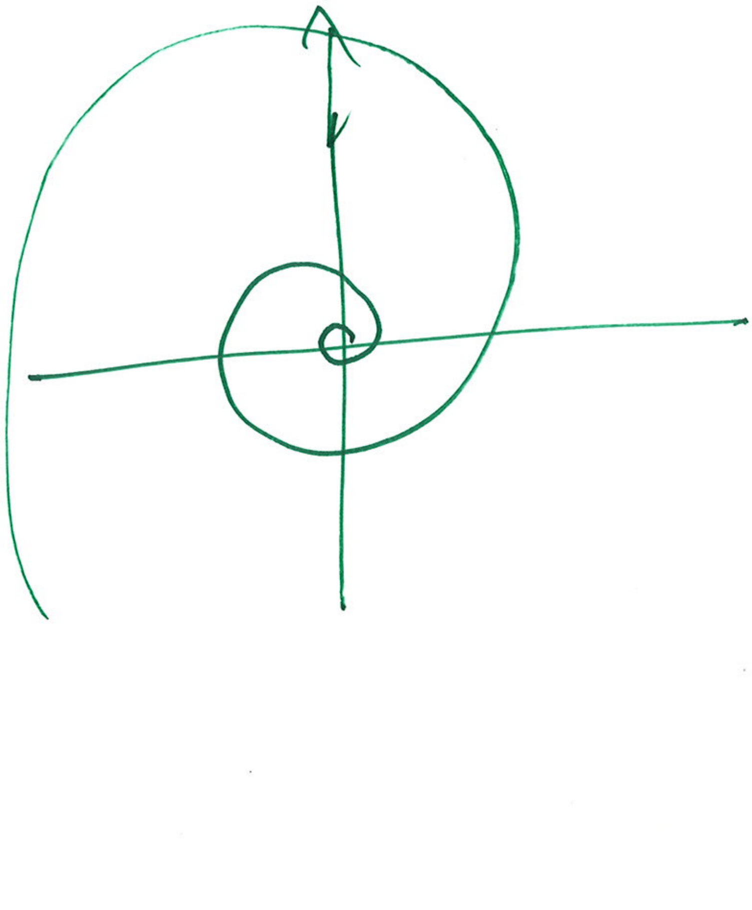
$0 < e < 1$: ellipse

$e = 1$: parabola

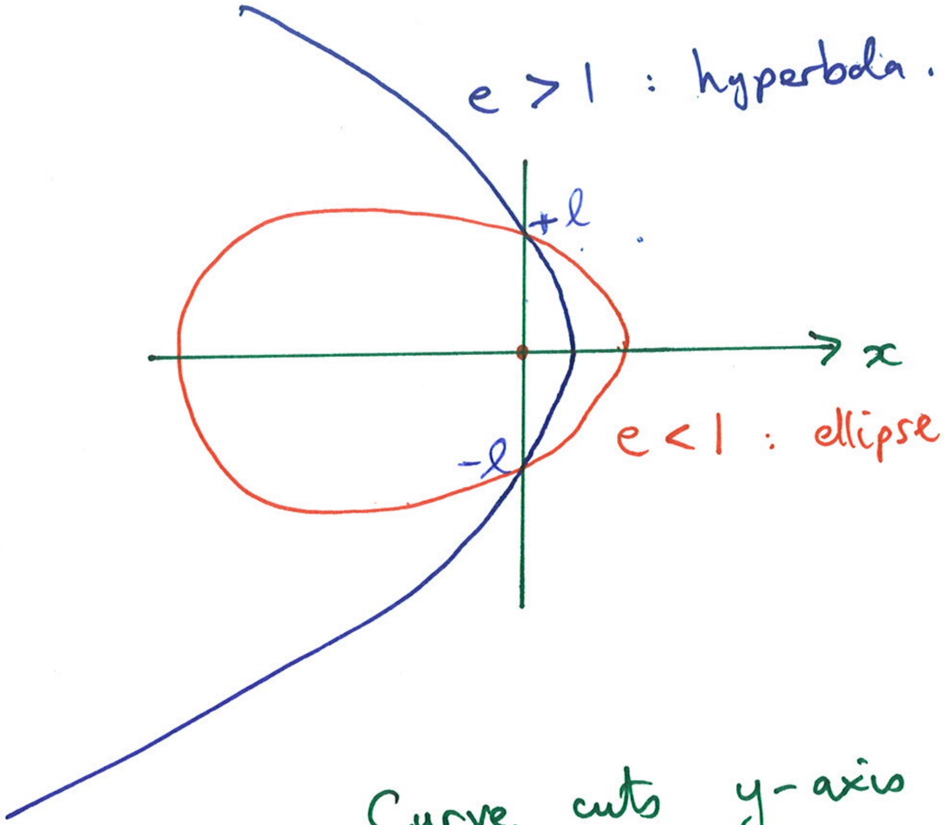
$e > 1$: hyperbola.



Archimedes spiral.



Log. spiral.



Curve cuts y-axis at $\pm l$.

Origin is ~~an~~ a focus, not the centroid (unless $e = 0$).

2.3.2 ARC-LENGTH IN POLAR COORDS.

Given $r = f(\theta)$,

we convert back to Cartesians and get

$$x = f(\theta)\cos\theta$$
$$y = f(\theta)\sin\theta$$

This looks like a parametric curve with θ as the parameter –

now we use Eq. 2.11 for arc-length

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

and

$$\frac{dx}{d\theta} = -f(\theta)\sin\theta + \frac{df}{d\theta}\cos\theta$$
$$\frac{dy}{d\theta} = f(\theta)\cos\theta + \frac{df}{d\theta}\sin\theta.$$

$$\Rightarrow L = \int_{\theta_1}^{\theta_2} \sqrt{[f(\theta)]^2 + \left(\frac{df}{d\theta}\right)^2}\, d\theta \qquad (2.27)$$

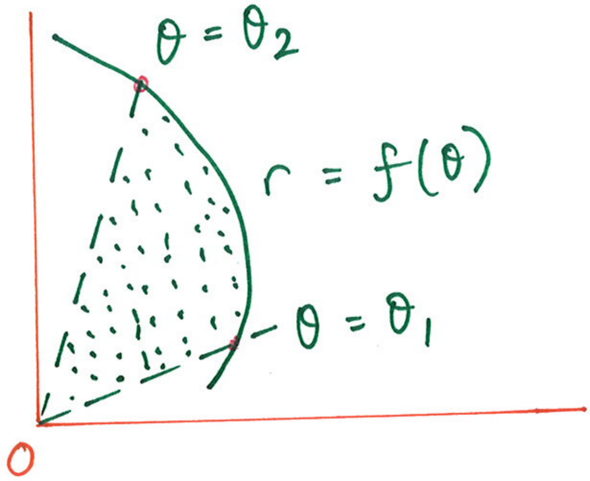
Note: this is different from 2.14.

2.3.3 AREA IN POLAR COORDS.

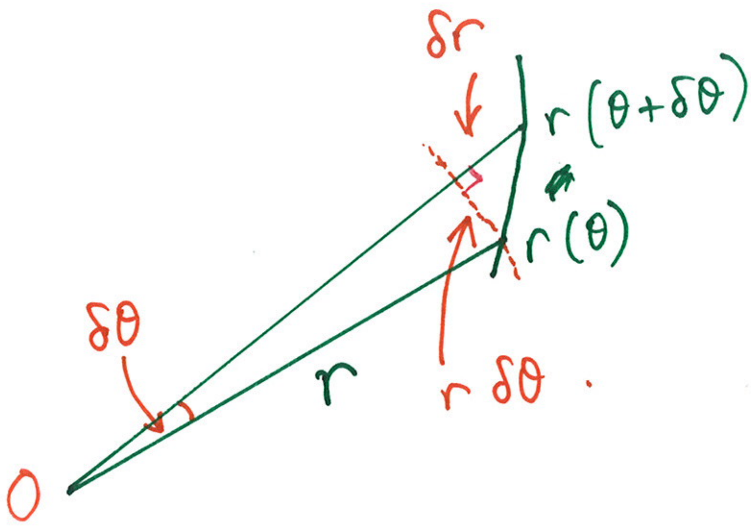
Given a polar curve

$$r = f(\theta),$$

suppose we want the area shown:

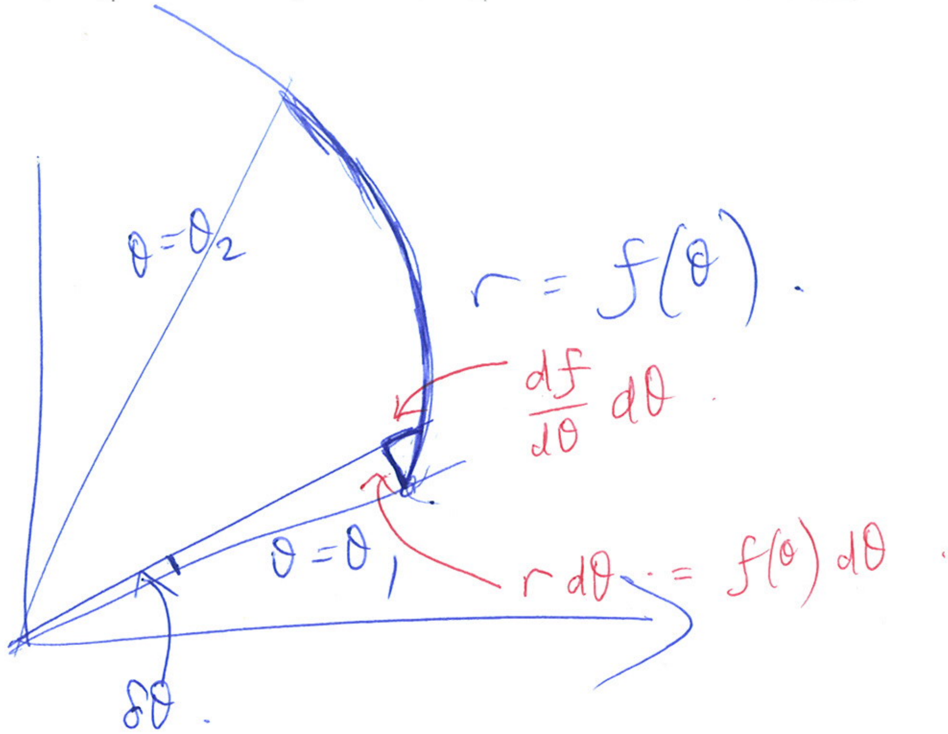


Consider a small interval from θ to $\theta + \delta\theta$:



As $\delta\theta \to 0$, area tends to isosceles triangle with length r and perp. base $r\,\delta\theta$

$$\Rightarrow \; dA = \frac{1}{2} r^2 \, d\theta \, .$$

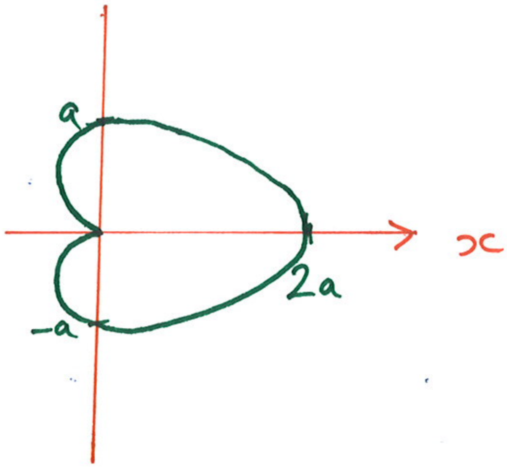


$\theta = \theta_2$
$r = f(\theta)$.
$\frac{df}{d\theta} d\theta$.
$\theta = \theta_1$
$r\, d\theta = f(\theta)\, d\theta$.
$\delta\theta$.

$$\Rightarrow \quad A = \frac{1}{2}\int_{\theta_1}^{\theta_2} \left[f(\theta)\right]^2 \, d\theta \qquad 2.28$$

Example 2.3:

'Cardioid' $\quad r = a(1+\cos\theta)$



$$\text{Area } A = \frac{1}{2}\int_0^{2\pi} \left[a(1+\cos\theta)\right]^2 \, d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta \quad d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} 1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta) \, d\theta$$

$$= \frac{a^2}{2}\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\left[2\pi + \pi\right] = \frac{3\pi a^2}{2}$$

2.4 PARAMETRIC SURFACES IN 3-D.

Previously we have met parametric curves in 2D and 3D,

$$\text{eg} \quad \underset{\sim}{r} = \underset{\sim}{r}(t) \quad \text{- one parameter } t.$$

The next step is to define a <u>surface</u> in 3D space.

A plane is the simplest type of surface, but we will generalise to curved surfaces.

A surface requires TWO parameters - we'll call these u, v so

$$x = f(u,v) \quad y = g(u,v) \quad z = h(u,v)$$

or $\quad \underset{\sim}{r} = \underset{\sim}{r}(u,v) \quad$ for short.

$$\underset{\sim}{r} = f(u,v)\,\underset{\sim}{i} + g(u,v)\,\underset{\sim}{j} + h(u,v)\,\underset{\sim}{k}.$$

We have seen $\underset{\sim}{r}(u, v_0)$ and $\underset{\sim}{r}(u_0, v)$ are lines on the surface.

From 2.2.1, $\dfrac{\partial \underset{\sim}{r}}{\partial u}(u, v_0)$

$$= \left(\frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u} \right) \quad \text{at } (u, v_0)$$

is the tangent vector to line $\underset{\sim}{r}(u, v_0)$.

Likewise $\dfrac{\partial \underset{\sim}{r}}{\partial v}(u_0, v)$ is the tangent vector to the line $\underset{\sim}{r}(u_0, v)$.

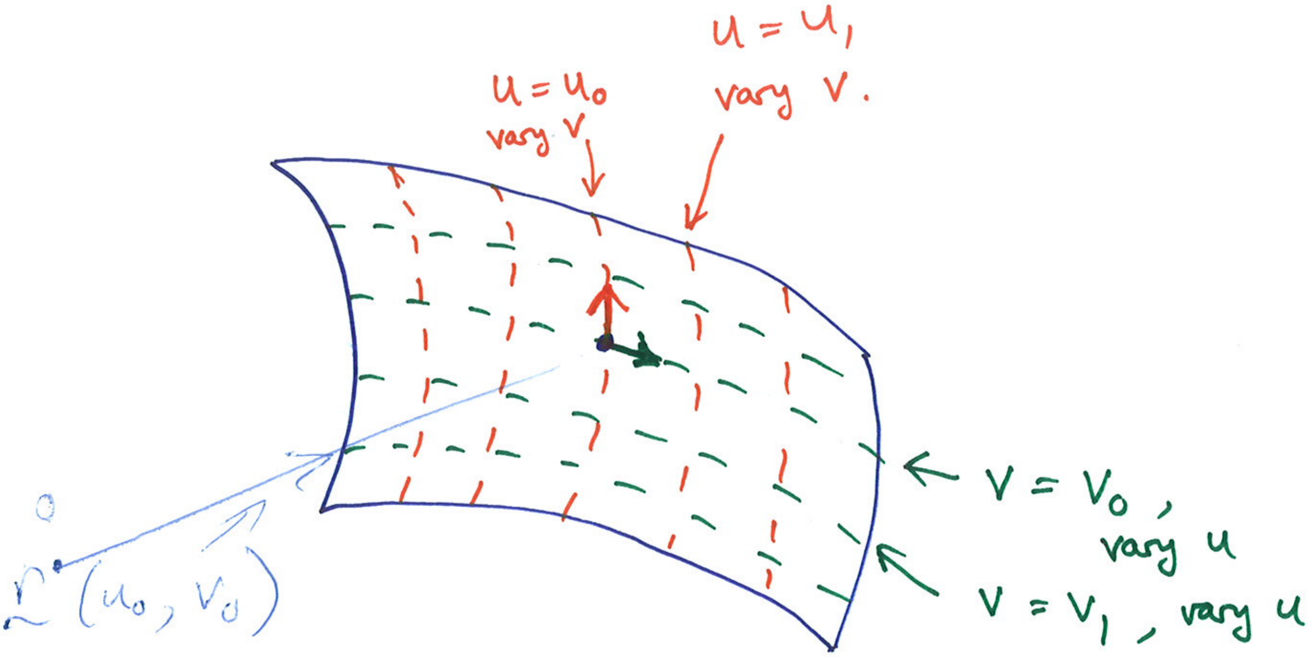
So both $\dfrac{\partial \underset{\sim}{r}}{\partial u}$ and $\dfrac{\partial \underset{\sim}{r}}{\partial v}$ are both tangent to the surface.

Now take

$$\underset{\sim}{N} = \frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v}$$

$\underset{\sim}{N} \neq \underset{\sim}{0}$ as long as both $\dfrac{\partial \underset{\sim}{r}}{\partial u}, \dfrac{\partial \underset{\sim}{r}}{\partial v}$ are non-zero and non-parallel.

then ~~so~~ $\underset{\sim}{N}$ is normal to the surface.



Fix $v = v_0$, vary u:

$\underset{\sim}{r} = \underset{\sim}{r}(u, v_0)$ is a line

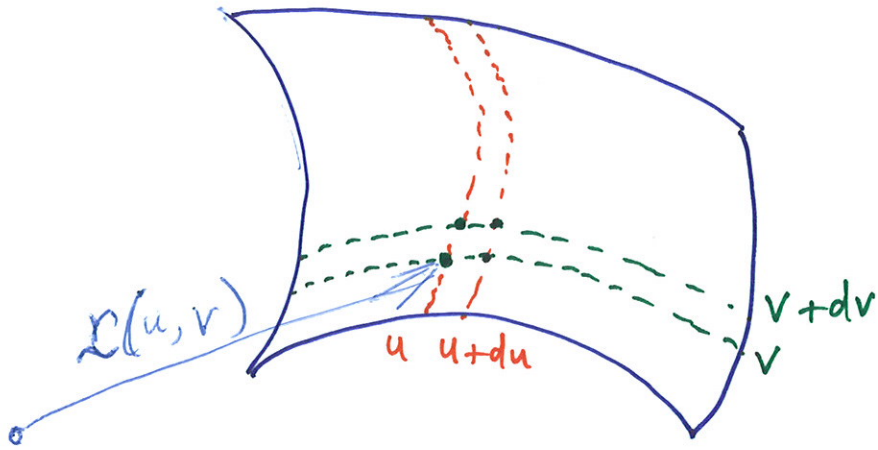
Fix $u = u_0$, vary v:

$\underset{\sim}{r} = \underset{\sim}{r}(u_0, v)$ is a different line

Lines cross at $\underset{\sim}{r}(u_0, v_0)$.

In general the <u>curved</u> surface is 'covered' with lines of constant u, v: imagine printing graph paper onto a balloon, then inflating it.

AREA OF A SURFACE :



Consider a point $\underset{\sim}{r}(u,v)$ in the surface,

and small changes du, dv :

the four points

$$\underset{\sim}{r}(u,v)$$
$$\underset{\sim}{r}(u+du,v)$$
$$\underset{\sim}{r}(u,v+dv)$$
$$\underset{\sim}{r}(u+du,v+dv)$$

make an infinitesimal <u>parallelogram</u>

in the surface. (ignoring second derivatives).

Two sides are

$$\underset{\sim}{r}(u+du,v) - \underset{\sim}{r}(u,v) = \frac{\partial \underset{\sim}{r}}{\partial u}\,du$$

and $$\underset{\sim}{r}(u,v+dv) - \underset{\sim}{r}(u,v) = \frac{\partial \underset{\sim}{r}}{\partial v}\,dv$$

Now we can find the equation for the tangent plane to our surface at point ~~$\underset{\sim}{r_0}$~~ $\underset{\sim}{r}_0 = \underset{\sim}{r}(u_0, v_0)$:

evaluate $\dfrac{\partial \underset{\sim}{r}}{\partial u}, \dfrac{\partial \underset{\sim}{r}}{\partial v}$ at (u_0, v_0)

evaluate cross-product $\underset{\sim}{N}_0 = \dfrac{\partial \underset{\sim}{r}}{\partial u} \times \dfrac{\partial \underset{\sim}{r}}{\partial v}$ at u_0, v_0.

Tan. plane is $(\underset{\sim}{r} - \underset{\sim}{r}_0) \cdot \underset{\sim}{N}_0 = 0$.

NOTE : if we're given point $\underset{\sim}{r}_0 = (x_0, y_0, z_0)$, first need to solve for u_0, v_0 giving that point.

The area dA of this parallelogram is the magnitude of the cross-product,

$$dA = \left| \frac{\partial \underset{\sim}{r}}{\partial u} du \times \frac{\partial \underset{\sim}{r}}{\partial v} dv \right|$$

$$= \left| \frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} \right| du\, dv .$$

Area of whole 'patch' ~~surface~~ is integral of this,

$$A = \iint_D \left| \frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} \right| du\, dv \qquad (2.34).$$

Limits on u, v set to 'cover' the given patch of the surface.

Special case : for a surface given by

$z = h(x,y)$ (explicit form)

just let $u = x$, $v = y$,

so $\underset{\sim}{r} = \cancel{(x, y, h}\ (u, v, h(u,v))$

Now $\dfrac{\partial \underset{\sim}{r}}{\partial u} = \left(1, 0, \dfrac{\partial h}{\partial u}\right)$

$\dfrac{\partial \underset{\sim}{r}}{\partial v} = \left(0, 1, \dfrac{\partial h}{\partial v}\right)$.

Cross-product is $\left(-\frac{\partial h}{\partial u}, -\frac{\partial h}{\partial v}, 1\right)$

so $$A = \iint_D \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1} \, dx \, dy. \qquad (2.35)$$

Example 2.4 :

Area of a sphere. radius a

A parametrisation of a sphere is

$$x = a \sin\theta \cos\phi$$
$$y = a \sin\theta \sin\phi$$
$$z = a \cos\theta$$

$$0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi$$

Here we've got θ, ϕ instead of u, v, (as in spherical polar coords - see later).

Easy to show $x^2 + y^2 + z^2 = a^2$.

Get: $$\frac{\partial \underset{\sim}{r}}{\partial \theta} = (a\cos\theta\cos\phi, \ a\cos\theta\sin\phi, \ -a\sin\theta)$$

and $$\frac{\partial \underset{\sim}{r}}{\partial \phi} = (-a\sin\theta\sin\phi, \ a\sin\theta\cos\phi, \ 0)$$

Cross product is

$$\frac{\partial \underset{\sim}{r}}{\partial \theta} \times \frac{\partial \underset{\sim}{r}}{\partial \phi} = \left(a^2 \sin^2\theta \cos\phi, \; a^2 \sin^2\theta \sin\phi, \; a^2 \cancel{\cos} \sin\theta \cos\theta \right)$$

([NB this = $a \sin\theta \, \underset{\sim}{r}$ - it' must be normal to the sphere so parallel to $\underset{\sim}{r}$]

and this has magnitude $a^2 \sin\theta$.

So area of sphere is

$$A = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} a^2 \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \left[-a^2 \cos\theta \right]_0^{\pi} d\phi$$

$$= \int_0^{2\pi} 2a^2 \, d\phi$$

$$= 4\pi a^2 .$$

2.4.1 PARAMETRIC FORMS OF COMMON SURFACES:

Plane: $\underset{\sim}{r}(u,v) = \underset{\sim}{r}_0 + u\underset{\sim}{a} + v\underset{\sim}{b}$

For $\underset{\sim}{r}_0$ any point in the plane,
$\underset{\sim}{a}$, $\underset{\sim}{b}$ two vectors parallel to plane.

Finite parallelogram:
as above with $0 \leq u \leq 1$,
$0 \leq v \leq 1$,

get parallelogram with one corner at $\underset{\sim}{r}_0$ and two sides $\underset{\sim}{a}$, $\underset{\sim}{b}$.

(so 4 corners are $\underset{\sim}{r}_0$, $\underset{\sim}{r}_0 + \underset{\sim}{a}$, $\underset{\sim}{r}_0 + \underset{\sim}{b}$, $\underset{\sim}{r}_0 + \underset{\sim}{a} + \underset{\sim}{b}$).

Cylinder: $(x, y, z) = (a\cos\theta, a\sin\theta, z)$

Params. θ, z.

$\rightarrow$ Cylinder, radius a, axis is z-axis

Sphere :

$$(x, y, z) = (a \sin\theta \cos\phi,\ a \sin\theta \sin\phi,$$
$$a \cos\theta)$$

Easy to check $x^2 + y^2 + z^2 = a^2$.

Ellipsoid :

$$(x, y, z) = (a \sin\theta \cos\phi,\ b \sin\theta \sin\phi,$$
$$c \cos\theta)$$

$$\rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Semi-axes are a, b, c.

Hyperboloid :

$$(x, y, z) = (a \cos u,\ b \sin u \cosh v,$$
$$c \sin u \sinh v)$$

$$\rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 .$$