Chapter 4

Vector integrals and integral theorems

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Syllabus covered:
1. Line, surface and volume integrals.
2. Vector and scalar forms of Divergence and Stokes's theorems. Conservative fields: equivalence to curl-free and existence of scalar potential. Green's theorem in the plane.

Calculus I and II covered integrals in one, two and three dimensional Euclidean (flat) space (i.e. $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$). We are still working in $\mathbb{R}^3$ so there is no generalization to be applied to volume or triple integrals, but we will generalise one dimensional integration from a straight line to an integral along a curve, and we will generalise two-dimensional integration from a region in a plane to a curved surface.

We will also be working with integration of vectors, though in many cases we will be using a scalar product so the final quantity to be integrated becomes a scalar. In the cases with a scalar product:

$\int f(x)\,\mathrm{d}x$ generalizes to $\int_{\mathscr{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ on a curve $\mathscr{C}$, called a **line integral** (section 4.1).
$\int\int f(x,y)\,\mathrm{d}x\mathrm{d}y$ generalizes to $\int_{\mathscr{S}} \mathbf{F}\cdot \mathrm{d}\mathbf{S}$ over a surface $\mathscr{S}$, called a **surface integral** (section 4.2).

We will then have to study the generalizations of

$$\int_a^b \frac{\mathrm{d}f}{\mathrm{d}x}\,dx = f(b) - f(a)\ , \tag{4.1}$$

called the 'fundamental theorem of calculus', which we use in the proofs. This theorem relates a one-dimensional integral to a (pair of) zero-dimensional evaluations at the two endpoints $x = a, b$. The higher dimensional versions do the following:

Stokes's theorem relates the surface integral of a curl to a line integral (2 dimensions to 1) around the edge of the surface: see section 4.6.

The Divergence Theorem[1] relates the volume integral of a divergence to a surface integral (3 dimensions to 2) over the boundary of the volume: see section 4.4.

There is also a special case of Stokes's theorem where the surface is a plane: this is Green's theorem

[1]First discovered by Joseph Louis Lagrange in 1762, then independently rediscovered by Carl Friedrich Gauss in 1813, by George Green in 1825 and in 1831 by Mikhail Vasilievich Ostrogradsky, who also gave the first proof of the theorem. Thus the result may be called Gauss's Theorem, Green's theorem, or Ostrogradsky's theorem.

relating the integral of a curl to a line integral (2 dimensions to 1): see section 4.5.

[Aside: All these are in fact special cases of the general Stokes's theorem which relates an $n-1$ dimensional integral of a field to the n dimensional integral of its derivative. Here the field is a generalization of a vector field called an $(n-1)$-form field.]

Finally we will discuss the application to potentials, and the proofs.

Before moving on to line and surface integrals, we consider the case where one wants to integrate a vector function $\mathbf{F}(u)$ of one variable, u, with respect to u. The integral can be calculated simply by integrating the components (in Cartesian coordinates) of $\mathbf{F} = (F_1, F_2, F_3)$:

$$\int \mathbf{F}\, \mathrm{d}u = \left(\int F_1\, \mathrm{d}u, \int F_2\, \mathrm{d}u, \int F_3\, \mathrm{d}u \right) \tag{4.2}$$

$$= \mathbf{i} \int F_1\, \mathrm{d}u + \mathbf{j} \int F_2\, \mathrm{d}u + \mathbf{k} \int F_3\, \mathrm{d}u \tag{4.3}$$

Integration of a vector in this case is just a set of three ordinary integrals. The restriction to Cartesian coordinates can be overcome by looking at the definition in vectorial terms: we go back to the basic definition of integration, which leads to a geometrical picture of $\mathbf{G} \equiv \int_a^b \mathbf{F} \mathrm{d}u$ (see Fig. 4.1):

$$\mathbf{G} = \int_a^b \mathbf{F}\, \mathrm{d}u = \lim_{\delta u_p \to 0} \sum_{p=1}^{N} \mathbf{F}(u) \delta u_p \; .$$

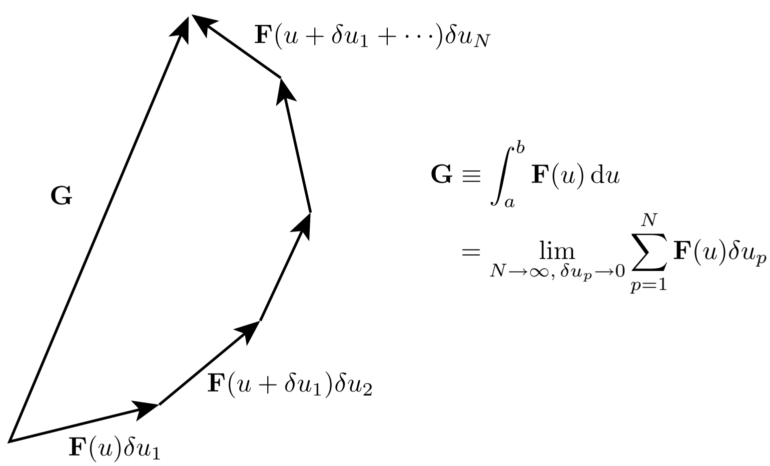

Figure 4.1: Geometrical picture of $\mathbf{G} = \int_a^b \mathbf{F}\, \mathrm{d}u = \lim_{\delta u_p \to 0} \sum_{p=1}^{N} \mathbf{F}(u) \delta u_p$.

Example 4.1. If $\mathbf{v}(t) \equiv \mathrm{d}\mathbf{r}/\mathrm{d}t$ is the velocity of a particle, as a function of time t, then

$$\int_{t_1}^{t_2} \mathbf{v}\, \mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\, \mathrm{d}t = \int_{\mathbf{r}(t_1)}^{\mathbf{r}(t_2)} \mathrm{d}\mathbf{r} = \mathbf{r}(t_2) - \mathbf{r}(t_1) \qquad .$$

Note here that $\mathbf{v}$ is the vector velocity of the particle, so the time-integral is the vector distance between the two end-points. If we had put v instead of $\mathbf{v}$ in the integral, then the result would be a scalar equal to the total arc-length of the curved path $\mathbf{r}(t)$, as we met in Chapter 2.

Warning: there seems to be a common belief that an integral always represents an area or volume. This comes from 1-D integration where $\int f(x)\, dx$ can be shown as an area between a curve $y = f(x)$ and the $x-$axis; or in 2-D integration the result $\int h(x,y)\, dxdy$ can be expressed as a volume between the xy plane and the surface $z = h(x,y)$. However, when we take general line and surface integrals the results are not necessarily areas and volumes; we will see that these integrals can represent various things such as distance travelled, work done by a force, flow of a fluid crossing a surface, etc, but there is not always a simple geometrical picture for the result of an integral.

4.1 Line Integrals

(See Thomas 16.1 and 16.2: note that Thomas begins by defining a scalar integral $\int f|\mathrm{d}\mathbf{r}|$, in the notation below. I come back to this at the end of this section.)
Suppose $\mathbf{F} = (F_1, F_2, F_3)$ is a vector field defined in some region of space, and $\mathscr{C}$ is a parametrized curve through that region from $\mathbf{r}_1$ to $\mathbf{r}_2$, so that $\mathscr{C}$ is given by

$$\mathbf{r}(t) = (g(t), h(t), q(t)) \qquad (t_1 \leq t \leq t_2),$$

and $\mathbf{r}_1 = \mathbf{r}(t_1)$, $\mathbf{r}_2 = \mathbf{r}(t_2)$. Then, one can define the line integral

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

to be

$$\int_{t_1}^{t_2} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t \quad \equiv \quad \int_{t_1}^{t_2} \left(F_1 \frac{\mathrm{d}g}{\mathrm{d}t} + F_2 \frac{\mathrm{d}h}{\mathrm{d}t} + F_3 \frac{\mathrm{d}q}{\mathrm{d}t} \right) \mathrm{d}t \tag{4.4}$$

Warning: do not forget to write the components of $\mathbf{F}$ in terms of the parameter t, so that t is the only variable that appears inside the integral!. Hence you must write $\mathbf{F}(\mathbf{r}) = \mathbf{F}(\mathbf{r}(t))$, so we replace $F_1(x, y, z)$ by $F_1(g(t), h(t), q(t))$, and so on; then we evaluate the dot product of $\mathbf{F}$ and $d\mathbf{r}/dt$, before finally integrating over t to get the numerical answer.

Second warning: it seems to be easy to confuse where one has to use $\mathbf{r}(t)$ and where one uses $\mathrm{d}\mathbf{r}/\mathrm{d}t$; you have to evaluate $\mathbf{F}$ at position $\mathbf{r}(t)$, while the line-segment $d\mathbf{r}$ is given by $(d\mathbf{r}/dt)\, dt$.

The above is just a version of the fundamental definition of an integral as the limit of lots of small contributions. In this case it's the scalar products of $\mathbf{F}(\mathbf{r})$ with small displacements $d\mathbf{r}$ along $\mathscr{C}$:

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \lim_{\delta \mathbf{r}_p \to 0} \sum_{p=1}^{N} \mathbf{F}(\mathbf{r}) \cdot \delta \mathbf{r}_p$$

If we are given a geometrical description of the curve without a parametrization, we have to first **find a parametrisation** of the described curve to actually evaluate the integral. For lines, circles, ellipses and so we can use, for example, (1.33) and (1.18)–(1.20).

Example 4.2. Evaluate the integral $\int \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ for the vector field $\mathbf{F} = -4xy\mathbf{i} + 8y\mathbf{j} + 2\mathbf{k}$, from the origin to the point $(2, 4, 1)$ along the following three paths:

1. along the curve $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t\mathbf{k}$, $0 \leq t \leq 2$,
2. from the origin to $(2, 0, 0)$, then from there to $(2, 4, 0)$, then to $(2, 4, 1)$, along straight lines [Note that the answer will be the sum of the three parts: a path may have several pieces, providing the next one begins where the previous one ends.]
3. on the surface $4x^2 + y^2 = 32z$ along a line with constant y/x.

Note that only for the first of these do we have the parametrization given: in the second and third we'll have to make a parametrization from the definitions.

1. In this case we are given the parametrised curve $\mathbf{r}(t)$ as above, and from that we get $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{i} + 2t\mathbf{j} + \frac{1}{2}\mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t)) = -4(t)(t^2)\mathbf{i} + 8(t^2)\mathbf{j} + 2\mathbf{k}$. The final bits we need is the $t-$ values at the given endpoints $\mathbf{r}_1 = (0,0,0)$ and $\mathbf{r}_2 = (2,4,1)$; it is easy to see those are $t = 0$ and $t = 2$ (solve the easiest equation e.g. $x = t$, and plug that in to the other two to check). Putting things together, we have

$$\begin{aligned}
\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}.\mathrm{d}\mathbf{r} &= \int_{t=0}^{2} \mathbf{F}(\mathbf{r}(t)).\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int_0^2 (-4t^3\mathbf{i} + 8t^2\mathbf{j} + 2\mathbf{k}).(\mathbf{i} + 2t\mathbf{j} + \tfrac{1}{2}\mathbf{k})\,\mathrm{d}t \\
&= \int_0^2 \left[(-4t^3)(1) + (8t^2)(2t) + (2)(\tfrac{1}{2})\right]\,\mathrm{d}t \\
&= \int_0^2 (12t^3 + 1)\,\mathrm{d}t \\
&= \left[3t^4 + t\right]_0^2 = 48 + 2 = 50.
\end{aligned}$$

2. Now our given "curve" is three straight line segments joined end-to-end and we need parametrisations for each, separately.
   The first segment is from $(0, 0, 0)$ to $(2, 0, 0)$. The straight line is, from the general form of Eq. 1.33 for the case of a line joining $\mathbf{r}_1$ and $\mathbf{r}_2$, i.e. $\mathbf{r} = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1)$,

$$\mathbf{r} = \mathbf{0} + t(2\mathbf{i}), \quad 0 \le t \le 1,$$

   Here we could call $2t$ simply x, so

$$\mathbf{r} = x\mathbf{i}, \quad 0 \le x \le 2.$$

   Along this line we have $\mathrm{d}\mathbf{r} = \mathbf{i}\,\mathrm{d}x$. To get the value of $\mathbf{F}$ we substitute $y = z = 0$ into the general form for $\mathbf{F}$, giving $\mathbf{F} = 2\mathbf{k}$. Taking the scalar product, $\mathbf{F}\cdot\mathrm{d}\mathbf{r} = 0$ and hence this segment gives a zero integral.

   In the second segment, from $(2, 0, 0)$ to $(2, 4, 0)$, we similarly get

$$\mathbf{r} = 2\mathbf{i} + y\mathbf{j}, \quad 0 \le y \le 4$$

   so along it, $\mathrm{d}\mathbf{r} = \mathbf{j}\mathrm{d}y$. Substituting $x = 2$, $z = 0$ in $\mathbf{F}$ we have $\mathbf{F} = -8y\mathbf{i} + 8y\mathbf{j} + 2\mathbf{k}$. So $\mathbf{F}\cdot\mathrm{d}\mathbf{r} = 8y\,\mathrm{d}y$ and hence this gives

$$\int_0^4 8y\,\mathrm{d}y = [4y^2]_0^4 = 64$$

   In the last segment, from $(2, 4, 0)$ to $(2, 4, 1)$,

$$\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} + z\mathbf{k}, \quad 0 \le z \le 1$$

   so along it we have $\mathrm{d}\mathbf{r} = \mathbf{k}\,\mathrm{d}z$, and $x = 2$, $y = 4$ gives $\mathbf{F} = -32\mathbf{i} + 32\mathbf{j} + 2\mathbf{k}$, so $\mathbf{F}\cdot\mathrm{d}\mathbf{r} = 2\,\mathrm{d}z$ and hence this gives $\int_0^1 2\,\mathrm{d}z = 2$.

   Finally adding the integrals from the three segments together, we get the full line integral over our given path $= 0 + 64 + 2 = 66$.

3. Now we are integrating along a line in a curved surface; the equation for the line is not given explicitly, but we are told two things which let us solve for it: the line is in the surface $4x^2 + y^2 = 32z$, and our line has constant y/x so $y = kx$ for some constant k. Geometrically, our line will be the intersection of a plane $y = kx$ (containing the $z-$axis) with the above surface. Since at the second end point $x = 2$ and $y = 4$, we need $k = 2$ so $y = 2x$. Substituting that in $4x^2 + y^2 = 32z$ gives $8x^2 = 32z$ so $x = 2\sqrt{z}$, and

$y = 4\sqrt{z}$. Now we have both x and y in terms of z, so we can use z as the one parameter for our curve: we have

$$\mathbf{r} = 2\sqrt{z}\mathbf{i} + 4\sqrt{z}\mathbf{j} + z\mathbf{k} \qquad 0 \le z \le 1 \quad ,$$

where the limits on z follow from the given endpoints. Once we have a one-parameter expression for the curve $\mathbf{r}(z)$, it is straightforward: we get $\mathrm{d}\mathbf{r} = (\mathbf{i}/\sqrt{z} + 2\mathbf{j}/\sqrt{z} + \mathbf{k})\mathrm{d}z$, while $\mathbf{F}(\mathbf{r}(z)) = -32z\mathbf{i} + 32\sqrt{z}\mathbf{j} + 2\mathbf{k}$. Hence inserting those into Eq. 4.4, remembering to take the scalar product, we have

$$\int_0^1 (-32\sqrt{z} + 64 + 2)\,\mathrm{d}z = \int_0^1 (66 - 32\sqrt{z})\,\mathrm{d}z = [66z - 64z^{3/2}/3]_0^1 = 66 - \frac{64}{3} \quad ,$$

which we could also write as $44\frac{2}{3}$.

Note: in the above, we could alternatively have chosen x as the one parameter, and write $y = 2x$, $z = x^2/4$ to get $\mathbf{r} = x\mathbf{i} + 2x\mathbf{j} + (x^2/4)\mathbf{k}$, and range $0 \le x \le 2$. It is straightforward to check that this gives the same result $44\frac{2}{3}$ for the line integral.

As well as giving some examples of how to calculate line integrals, this example makes the important point that in general the result depends on the curve, not just on its two endpoints. We shall return to this matter in Section 4.7, where we will find that if $\mathbf{F}$ has zero curl (irrotational), the resulting line integral only depends on the two endpoints, not the curve between them.

Exercise 4.1. Calculate $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$, where $\mathbf{F} = 4yz\mathbf{i} - 3z\mathbf{j} + 2x^2\mathbf{k}$, over each of the following curves from $(0,0,0)$ to $(1,1,1)$:

(a) C: $\quad \mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \quad 0 \le t \le 1$

(b) C: $\quad \mathbf{r} = t^2\mathbf{i} + t\mathbf{j} + t^3\mathbf{k} \quad 0 \le t \le 1$

□

If the vector field $\mathbf{F}$ represents a **force** (e.g. gravitational force), then

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}.\mathrm{d}\mathbf{r}$$

is called a **work integral** and its value is the **work done** by the force for a particle moving between $\mathbf{r}_1$ and $\mathbf{r}_2$, which equals the increase in energy of the body acted on. This occurs because for each small movement $d\mathbf{r}$, (small enough to be a straight line), if θ is the local angle between $\mathbf{F}$ and $d\mathbf{r}$, then $F\cos\theta$ is the component of force *parallel* to $d\mathbf{r}$, so $\mathbf{F}\cdot d\mathbf{r} = F\,dr\cos\theta$ is the work done by the force, along the small step $d\mathbf{r}$. The line integral just adds up that work along all the small steps along the path, so the line integral is the total work done from $\mathbf{r}_1$ to $\mathbf{r}_2$.

If instead of representing a force, $\mathbf{F}$ represents the velocity field in a fluid, and if $\mathscr{C}$ is some curve in the fluid, then $\int_{\mathscr{C}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ is called the **flow** along curve $\mathscr{C}$. If $\mathscr{C}$ is a closed curve, the flow is called the **circulation** around $\mathscr{C}$.

Finally, note that Thomas's form $\int f|\mathrm{d}\mathbf{r}|$ is obtained if one assumes that $\mathbf{F}$ is parallel to the unit tangent vector to the curve, $\mathbf{t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\Big/\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|$, at all points on the curve, since $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\mathrm{d}t = \mathbf{t}|\mathrm{d}\mathbf{r}|$, and in this case, taking $\mathbf{F} = f\mathbf{t}$,

$$\mathbf{F}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t = f\mathbf{t}\cdot\mathbf{t}|\mathrm{d}\mathbf{r}| = f|\mathrm{d}\mathbf{r}| \qquad .$$

Thus Thomas's starting point is simply a special case of the general line integral.

4.2 Surface integrals

(See Thomas 16.5 and 16.6, but be aware that Thomas starts by defining the integral of a scalar, using what is, in the notation below, $\int f|\mathrm{d}\mathbf{S}|$.)

To define surface integrals, we now have to take into account that a small area on a curved surface has both a magnitude and a direction (the normal to the surface) associated with it, so we can represent a small area as a vector, as we saw in Chapter 2.

Consider an area S in a plane (see Fig. 4.2a). If $\mathbf{n}$ is a unit vector perpendicular to the plane, then the vector representing the area, $\mathbf{S}$, is defined to be

$$\mathbf{S} = S\mathbf{n}$$

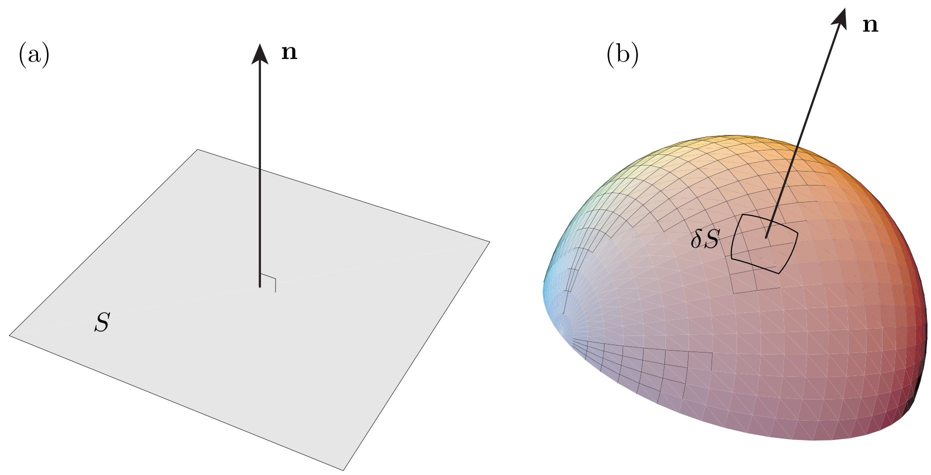

Figure 4.2: (a) Normal $\mathbf{n}$ to a plane area S. The vector area is $\mathbf{S} = S\mathbf{n}$. (b) Normal $\mathbf{n}$ to a more general surface. The vector area of the small surface element is $\delta\mathbf{S} = \delta S\,\mathbf{n}$, where δS is the magnitude of the area.

In the case of a curved surface in three dimensions (see 4.2b), we need to pick a small area δS which is small enough to be approximated as (almost) flat, and define the vector $\delta\mathbf{S}$ for that area element δS as

$$\delta\mathbf{S} = \delta S\,\mathbf{n} \qquad ,$$

where $\mathbf{n}$ is a unit vector normal to the surface element δS. Note we are still using the convention that vectors are written in bold type and the same symbol in ordinary type means the magnitude, thus $\delta S = |\delta\mathbf{S}|$. In the limit we shall write $\mathrm{d}S$ rather than δS. (Thomas uses $\mathrm{d}\sigma$ for this $\mathrm{d}S$.)

Note we still have a sign ambiguity in this definition, because either direction of the unit vector along the normal line could be used. One case where we can fix the sign is the case of a **closed** surface, where $\mathbf{n}$ is generally taken to be the **outward-pointing** unit normal vector. If the surface is not closed, we will have to explicitly specify geometrically one of the two possible directions for $\mathbf{n}$.

Now that we have defined how to represent a small area as a vector, we can now define the **surface integral** for a vector field $\mathbf{F}$ over a general curved surface $\mathscr{S}$:

$$\int_{\mathscr{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{S} \quad = \quad \int_{\mathscr{S}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S \qquad . \tag{4.5}$$

Such an integral is also called the **flux** of $\mathbf{F}$ across area $\mathscr{S}$. Since the quantity integrated is a scalar product of two vectors, the answer is a scalar quantity.

These surface integrals arise in a number of physical situations: one example is the case where $\mathbf{F}$ represents the velocity field in a fluid, where the surface integral represents the volume of fluid crossing the surface $\mathscr{S}$ per unit time. Another example is if $\mathbf{F}$ is a magnetic field $\mathbf{B}$, in which case the integral would be the *magnetic flux* across the surface $\mathscr{S}$. (These results occur because $F\cos\theta$ is the component of $\mathbf{F}$ parallel to the local normal $\mathbf{n}$ i.e. perpendicular to the surface; while the component of $\mathbf{F}$ parallel to the surface (perpendicular to $\mathbf{n}$) does not contribute to the flux across the surface. Thus, the flux of $\mathbf{F}$ crossing any small patch of surface $d\mathbf{S}$ is $|\mathbf{F}|\cos\theta\, dS$, which is $\mathbf{F}\cdot d\mathbf{S}$ from the definition of the dot product. The integration then just adds up the contribution from all the infinitesimal patches, to get the flux crossing the whole curved surface.

The double integrals in a plane that we met before, $\int\int f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, can be thought of as integrals of $\mathbf{F}.\mathrm{d}\mathbf{S}$, where
$\mathbf{F} = f\mathbf{k}$ and $\mathrm{d}\mathbf{S} = (\mathrm{d}x\,\mathrm{d}y)\mathbf{k}$.

The tricky part is, once we are given a field $\mathbf{F}$ and a surface $\mathscr{S}$, to turn the general form $\int_{\mathscr{S}}\mathbf{F}.\mathrm{d}\mathbf{S}$ into a double integral that we can actually do. We shall give some general rules after studying some examples.

We next look at three examples of increasing difficulty: one is a simple plane case, the second a curved surface where the integral is easy, and the third gives us the patterns we need for the general case.

Example 4.3. If $\mathbf{F} = (3x,\, 2xz,\, 3)$, evaluate the flux of $\mathbf{F}$ across the surface $\mathscr{S}$: $z = 0,\, 0 \le x \le 1,\ \ 0 \le y \le 2$ (where the normal is to be in the positive z direction).

Here the given surface is a rectangle in the xy-plane, so the normal $\mathbf{n}$ is $\pm\mathbf{k}$. We are told to take the plus sign. We need to integrate over x and y with limits as above:

$$\int_{\mathscr{S}}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \int_0^1\int_0^2(3x\mathbf{i}+(2x)0\mathbf{j}+3\mathbf{k}).(0\mathbf{i}+0\mathbf{j}+1\mathbf{k})\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^1\int_{y=0}^2 3\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 6\,\mathrm{d}x = 6.$$

Example 4.4. If the velocity field of a fluid is $\mathbf{v} = \frac{1}{r^2}\mathbf{e}_r$, where r is the distance from the origin O and $\mathbf{e}_r$ is a unit vector at position $\mathbf{r}$ pointing away from the origin, find the flux $\int\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S$ across a sphere $\mathscr{S}$ of radius a whose centre is at the origin. (The outward normal should be taken.)

In this case, the outward normal and $\mathbf{e}_r$ are the same vector, so

$$\mathbf{v}\cdot\mathbf{n} = \frac{1}{r^2}\mathbf{e}_r\cdot\mathbf{e}_r = \frac{1}{r^2}$$

($\mathbf{e}_r.\mathbf{e}_r = 1$ because $\mathbf{e}_r$ is a unit vector). On the given sphere of radius a, $r = a$, so

$$\int_{\mathscr{S}}\mathbf{v}.\mathbf{n}\,\mathrm{d}S = \int_{\mathscr{S}}\frac{1}{a^2}\,\mathrm{d}S = \frac{1}{a^2}\times(\text{Area of sphere of radius } a) = \frac{1}{a^2}4\pi a^2 = 4\pi \quad .$$

using the fact that $\frac{1}{a^2}$ is a constant, so can be taken outside the integral sign.

Example 4.5. Find the flux of the field $\mathbf{F} = z\mathbf{k}$ across the portion of the sphere $x^2+y^2+z^2 = a^2$ in the first octant (this is the $1/8$-th of space in which x, y and z are all ≥ 0) with normal taken in the direction away from the origin.

This example is easier in spherical polars (see later), but we can do it in Cartesians. Write the required part of the sphere as a surface $z = \sqrt{a^2-x^2-y^2}$ (note that for a whole sphere we would also need the points where $z = -\sqrt{a^2-x^2-y^2}$, the square root being understood to be the non-negative one). Consider the displacement vector for a small change $\mathrm{d}x$, by taking the derivative of $\mathbf{r} = (x,\, y,\, \sqrt{a^2-x^2-y^2})$ as in section

3.1. It will be

$$\frac{\partial \mathbf{r}}{\partial x}\mathrm{d}x = \left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}\right)\mathrm{d}x = \left(1,\, 0,\, \frac{-x}{\sqrt{a^2-x^2-y^2}}\right)\mathrm{d}x \tag{4.6}$$

and similarly a small change in y gives a displacement

$$\frac{\partial \mathbf{r}}{\partial y}\mathrm{d}y = \left(0,\, 1,\, \frac{-y}{\sqrt{a^2-x^2-y^2}}\right)\mathrm{d}y\ . \tag{4.7}$$

The magnitude of the corresponding area element is then given by the area of a parallellogram with sides (4.6) and (4.7), and the normal direction is perpendicular to them both, so we need their cross-product

$$\begin{aligned}
\mathrm{d}\mathbf{S} &= \left(1,\, 0,\, \frac{-x}{\sqrt{a^2-x^2-y^2}}\right)\mathrm{d}x \times \left(0,\, 1,\, \frac{-y}{\sqrt{a^2-x^2-y^2}}\right)\mathrm{d}y \\
&= \left(\frac{x}{\sqrt{a^2-x^2-y^2}}\mathbf{i} + \frac{y}{\sqrt{a^2-x^2-y^2}}\mathbf{j} + \mathbf{k}\right)\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

Thus $\mathbf{F}\cdot\mathrm{d}\mathbf{S} = z\,\mathrm{d}x\,\mathrm{d}y = \sqrt{a^2-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y$.

Now we need the limits on the variables. The first octant of the sphere lies above the first quadrant of the circle $x^2+y^2=a^2$, $z=0$, so we will have

$$\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,\mathrm{d}y\,\mathrm{d}x\ .$$

The rest of the problem is just a double integral like those in Calculus II. We can do it by a substitution such as $y = \sqrt{a^2-x^2}\sin\xi$ which gives

$$\int_0^a (a^2-x^2)\int_{\xi=0}^{\pi/2}\cos^2\xi\ \mathrm{d}\xi\,\mathrm{d}x$$

and this turns out to be $\pi a^3/6$ using the double-angle formula.

Note that parametrization by a pair of coordinates will not always give all the surface: for example, consider the surface consisting of two touching perpendicular squares, one square with a vertex at the origin and sides 1 along the x and y axes, and the similar square in the (x,z) plane: this surface cannot be covered by any pair of the Cartesian coordinates, though it can easily be split into two pieces each of which separately can be handled that way, and the results added.

The final part of the above example provides general methods for turning a surface integral like Eq. 4.5 into a double integral we can actually do. We next look at 3 cases:

1. Surface given by two parameters $\mathbf{r}(u,v)$.
2. Surface given by $z = h(x,y)$
3. Surface given by $g(x,y,z) =$const.

Note that if we are only given a geometrical "description" of the surface, we will need to put our surface into one of the above forms before we proceed: which is easiest may depend on the surface, but usually the two-parameter case is simplest.

4.2.1 Surface integral: surface given by two parameters

First consider the case where *the surface is given, or can be found, in terms of two parameters*: several examples were covered in Chapter 2. See also Thomas 16.6, and diagrams 16.55 and 16.56. For a surface given by two parameters u,v we have:

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}\ .$$

Now we can do the surface integral as follows:

1. Calculate the partial derivatives $\dfrac{\partial \mathbf{r}}{\partial u}$ and $\dfrac{\partial \mathbf{r}}{\partial v}$.
2. Calculate the cross product
$$\mathrm{d}\mathbf{S} = \left(\frac{\partial \mathbf{r}}{\partial u}\right) \times \left(\frac{\partial \mathbf{r}}{\partial v}\right) du\, dv$$
As we showed previously, this vector is normal to the surface and has magnitude equal to the area of the small parallelogram with four corners given by $\mathbf{r}(u,v)$, $\mathbf{r}(u+du,v)$, $\mathbf{r}(u,v+dv)$, $\mathbf{r}(u+du,v+dv)$, so it is the $d\mathbf{S}$ we want.
3. Express $\mathbf{F}$ in terms of u,v using $\mathbf{r} = \mathbf{r}(u,v)$ as given above and substituting.
4. Form the scalar product $\mathbf{F}\cdot \mathrm{d}\mathbf{S}$
5. From the given geometry of the surface, work out appropriate limits on u,v and perform the double integral over du and dv.

This gives us finally

$$\int_{\mathscr{S}} \mathbf{F}\cdot d\mathbf{S} = \int_v \int_u \mathbf{F}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) du\, dv$$

Warning: note that the cross-product above may be opposite to the required normal direction, so one may need to take its negative (which is equivalent to just swapping the order in the cross-product).
Both for this reason, and for working out limits on the variables, it is a good idea to draw a sketch first.

For the standard surfaces such as cylinders, spheres and ellipsoids we already know some parametrizations, (1.28)–(1.31).

4.2.2 Surface integral: surface $z = h(x,y)$

The second case to consider is where we have a surface given as one coordinate is a function of the other two, e.g. $z = h(x,y)$. This is essentially a special case of the more general two-parameter case above where $x = u,\ y = v,\ z = h(u,v)$. Just using x and y as the parameters, we get the surface as $\mathbf{r} = (x,y,\,h(x,y))$, and partial differentiation gives

$$\frac{\partial \mathbf{r}}{\partial x} = (1, 0, \partial h/\partial x) \qquad \frac{\partial \mathbf{r}}{\partial y} = (0, 1, \partial h/\partial y) \qquad ;$$

so the area element on the curved surface $z = h(x,y)$ is again the cross product of the above, which is

$$\mathrm{d}\mathbf{S} = (-\partial h/\partial x,\, -\partial h/\partial y,\, 1)\ \mathrm{d}x\,\mathrm{d}y \qquad .$$

Next we evaluate $\mathbf{F}(\mathbf{r})$ on the surface using $\mathbf{r} = (x,y,h(x,y))$ again, we evaluate the scalar product $\mathbf{F}\cdot d\mathbf{S}$, and finally do the double integral with respect to x,y.

(There are other similar cases if instead x is given as a function of y,z by $x = g(y,z)$; this is very similar to the above except for swapping x,y,z) .

Aside: It is also useful to note that the unit normal to the surface $z = h(x,y)$ is

$$\mathbf{n} = \frac{1}{\sqrt{(\partial h/\partial x)^2 + (\partial h/\partial y)^2 + 1}}(-\partial h/\partial x, -\partial h/\partial y, 1)$$

Since $\mathbf{n}$ is a unit vector, the angle θ this makes with the z axis is given by

$$\cos\theta = \mathbf{k}.\mathbf{n} = 1/\sqrt{(\partial h/\partial x)^2 + (\partial h/\partial y)^2 + 1} \qquad .$$

The magnitude $\mathrm{d}S = |\mathrm{d}\mathbf{S}|$ is then
$dS = \sqrt{(\partial h/\partial x)^2 + (\partial h/\partial y)^2 + 1}\,\mathrm{d}x\,\mathrm{d}y = \mathrm{d}x\,\mathrm{d}y/\cos\theta$.

This is not needed for the surface integral in the current case, but we will make use of this result in the next section .

4.2.3 Surface given by $g(x,y,z) = constant$

The third case of a surface integral is that where we are given a vector field $\mathbf{F}$, and where our surface is defined by a function $g(x,y,z) = \text{const}$, (and some specified boundaries), when we do not necessarily have a convenient parametrization. As long as the surface is single-valued in two coordinates, e.g. for a given x,y there is a unique z on the surface, we can use those two coordinates e.g. x,y as the two parameters as follows:

1. Calculate ∇g (which is the vector normal to the surface).
2. Find the unit normal vector in that direction $\mathbf{n} = \nabla g/|\nabla g|$.
3. Calculate $\cos\theta = \mathbf{n}\cdot\mathbf{k}$, where θ is the angle between $\mathbf{n}$ and the $+z$-direction.
4. Write $\mathrm{d}\mathbf{S} = \mathbf{n}\,\mathrm{d}S = \mathbf{n}\,\mathrm{d}x\,\mathrm{d}y/\cos\theta$, using the result from the previous subsection. (For a geometrical illustration, consider a 'light bulb' at $z = +\infty$. A small patch on our surface with area dS would cast a 'shadow' of area $dS\cos\theta$ on the xy plane; reversing this, the required area dS on the surface which casts a shadow of area $dx\,dy$ will be $dS = dx\,dy/\cos\theta$).

   Combining the above expressions for $\mathbf{n}$ and $\cos\theta$ gets us $\mathrm{d}\mathbf{S} = (\nabla g)\mathrm{d}x\,\mathrm{d}y/(\nabla g\cdot\mathbf{k})$.
5. Finally, use this to form $\mathbf{F}.\mathrm{d}\mathbf{S}$, and do the double integration with respect to x and y.

Thus, we can use (x, y) as our two parameters, provided $\cos\theta \neq 0$ over our range of x,y, and also provided that we can express $\mathbf{F}(\mathbf{r})$ on the surface in terms of x and y. Here we may need to solve for z in terms of x,y on our given surface; or if we are lucky, things may simplify so that at given x, y and $g(x,y,z)$ we can evaluate $\mathbf{F}.d\mathbf{S}$ without actually needing to solve for z.

Note that Thomas gives an even more general version of this where he considers a plane with normal $\mathbf{p}$ and an area $\mathrm{d}A$ in the plane (in place of $\mathbf{k}$ and $\mathrm{d}x\,\mathrm{d}y$): because he is working with $|\mathrm{d}S|$ he uses $|\cos\theta|$ and writes $1/|\cos\theta|$ as $|\nabla g|/|\nabla g.\mathbf{p}|$. While one is unlikely to need to use a general $\mathbf{p}$, that version has the advantage of covering the three cases $\mathbf{p} = \mathbf{i}$, $\mathbf{p} = \mathbf{j}$ and $\mathbf{p} = \mathbf{k}$ in one formula.

Exercise 4.2. If $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, evaluate

$$\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$$

where S is the rectangular box formed by the six planes

$$x = 0, a, \quad y = 0, b, \quad z = 0, c.$$

□

Exercise 4.3. If $\mathbf{F} = 3y^2\mathbf{i} - \mathbf{j} + xz\mathbf{k}$, evaluate the integral $\int_{\mathscr{S}} \mathbf{F}.\mathrm{d}\mathbf{S}$, where $\mathscr{S}$ is the surface $z = 1$, $0 \le x \le 1$, $0 \le y \le x$ (take the normal pointing in the positive z direction).
[Answer: $1/3$] □

Exercise 4.4. If $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, evaluate

$$\int_S \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S$$

over the hemispherical surface S given by $z \ge 0$, $x^2 + y^2 + z^2 = a^2$, taking the normal outward from the origin.
[Answer: πa^2] □

To link up with Thomas, his initial $\int f|\mathrm{d}\mathbf{S}|$ is just $\int \mathbf{F}.\mathrm{d}\mathbf{S}$ for a vector field such that $\mathbf{F} = f\mathbf{n}$ on the surface.

4.3 Volume Integrals

In Cartesian coordinates, consider a small cuboid with one corner at (x,y,z) and sides (dx,dy,dz). This has the eight corners (x,y,z), $(x+dx,y,z)$, $\ldots$, $(x+dx,y+dy,z+dz)$, and the infinitesimal volume of the cuboid is obviously $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Since in this course we will not be considering curved three-dimensional objects in four-dimensional space, we do not have to think about a vectorial version of dV.

However, the fact that dV is a *volume* element is an important way to look at it. If we re-label our space using new coordinates (u, v, w), then taking small displacements du, dv, dw gives us small displacements $(\partial\mathbf{r}/\partial u)du$, $(\partial\mathbf{r}/\partial v)dv$, $(\partial\mathbf{r}/\partial w)dw$ in ordinary x,y,z space. These three vectors will form a small parallelepiped, and the volume of that parallelepiped dV is given by a scalar triple product of the three vectors above (see section 1.7); that will give the Jacobian determinant for change of variables in a triple integral,

$$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

as in section 1.3; so this explains why the Jacobian formula works.

Usually the integrand of a volume integral is a scalar. However, we could integrate vectors in $\mathbb{R}^3$, though this is not so often used. Given a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, one can define

$$\int_V \mathbf{F}\,\mathrm{d}V = \left(\int_V F_1 \mathrm{d}V\right)\mathbf{i} + \left(\int_V F_2 \mathrm{d}V\right)\mathbf{j} + \left(\int_V F_3 \mathrm{d}V\right)\mathbf{k}$$

For example, $\mathbf{F}$ might be the momentum vector field in a fluid, (in that case we would have $\mathbf{F} = \rho\mathbf{v}$ where ρ is the density and $\mathbf{v}$ is the velocity); the volume integral above would then equal the total net momentum of that volume of fluid.

The most useful integrals we will deal with from here onwards are the line integral $\int_{\mathscr{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, the flux across a surface, $\int_{\mathscr{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, and the integral of a scalar over a volume, $\int_V f\,\mathrm{d}V$.

4.4 The Divergence Theorem

(See Thomas 16.8)
The Divergence Theorem states (following Thomas's wording) that "under suitable conditions":

Theorem 4.1 *The flux of a vector field* **F** *across a closed oriented surface* $\mathscr{S}$ *in the direction of the surface's outward unit normal vector field* **n** *equals the integral of* $\nabla\cdot\mathbf{F}$ *over the region* $\mathscr{D}$ *enclosed by the surface*

$$\int_{\mathscr{D}} \nabla\cdot\mathbf{F}\,\mathrm{d}V = \int_{\mathscr{S}} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S \equiv \int_{\mathscr{S}} \mathbf{F}\cdot\mathrm{d}\mathbf{S} \tag{4.8}$$

If asked to state this theorem, you **must** define the terms used, and state the conditions on the surface (i.e. "closed, oriented") and on the direction of the normal (outward).

We have not spelt out here in detail the 'suitable conditions' required of **F** and the surface. These, and a proof, are discussed in section 4.9, but will not be examinable.

Here the word 'Oriented' means we assign an outward direction for the normal to S in a consistent and continuous way. An S for which this is possible is called *orientable*: the Möbius strip (see Thomas Fig. 16.46) is an example of a non-orientable surface.

Note that it is not required that $\mathscr{S}$ has a single connected piece. For instance, it could have two parts, one inside the other, and then $\mathscr{D}$ would be the volume in between.

The Divergence theorem appears in a number of important physical situations such as Maxwell's equations in electromagnetism, and various cases in fluid dynamics. From a purely mathematical viewpoint, another use is that to calculate either of the integrals in it, we can use the other one if it is easier to do.

In the next example we calculate both sides of the Divergence Theorem for a simple case, and verify they really are equal.

Example 4.6. Suppose $f = xy$. Find a vector field **F** such that $\nabla\cdot\mathbf{F} = f$. Suppose V is the closed rectangular volume bounded by the planes $x = 0, a$, $y = 0, b$, $z = 0, c$, and $\mathscr{S}$ is the surface of the volume. Evaluate directly

$$\int_V f\,\mathrm{d}V \quad \text{and} \quad \int_{\mathscr{S}} \mathbf{F}.\mathbf{n}\,\mathrm{d}S$$

(where **n** is an outward normal), and show that they are equal – as they should be, according to the Divergence Theorem.

The volume integral is straightforward.

$$\begin{aligned}
\int_0^c\int_0^b\int_0^a xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^c\int_0^b [\tfrac{1}{2}x^2y]_0^a\,\mathrm{d}y\,\mathrm{d}z = \int_0^c\int_0^b \tfrac{1}{2}a^2y\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^c [\tfrac{1}{4}a^2y^2]_0^b\,\mathrm{d}z = \int_0^c \tfrac{1}{4}a^2b^2\,\mathrm{d}z = \tfrac{1}{4}a^2b^2[z]_0^c = \tfrac{1}{4}a^2b^2c.
\end{aligned}$$

There are numerous ways to construct a vector field **F** of the required form, e.g. by integrating f with respect to x and making this the x-component of a vector **F**, so

$$\mathbf{F} = \left(x^2y/2\,, 0, 0\right)\ .$$

Our closed surface $\mathscr{S}$ enclosing V is a cuboid with six faces, so we must evaluate $\mathbf{F}.\mathbf{n}$ on each of the six and add the results. Since our cuboid is aligned with the x, y, z axes, on two of the faces, $\mathbf{n} = \pm\mathbf{i}$, on two $\mathbf{n} = \pm\mathbf{j}$ and on the last two $\mathbf{n} = \pm\mathbf{k}$.

Because $\mathbf{F} \propto \mathbf{i}$ is always parallel to the x-direction, $\mathbf{F}.\mathbf{n} = 0$ on the four faces where $\mathbf{n} = \pm\mathbf{j}$, $\pm\mathbf{k}$, so those give zero surface integral. The remaining faces are the two where $x = 0$ and $x = a$:
On the $x = 0$ face, $\mathbf{F} = \mathbf{0}$ and so $\mathbf{F}.\mathbf{n} = 0$. This leaves only the face $x = a$. On that face $\mathbf{F}.\mathbf{n} = (a^2y/2)\mathbf{i}.\mathbf{i} =$

$a^2y/2$, and we have $dS = dydz$. Integrating this over that face with respect to y, z gives

$$\int_{\mathscr{S}} \mathbf{F}.\mathbf{n} = \int_0^b \int_0^c \frac{1}{2}a^2 y \,\mathrm{d}z\,\mathrm{d}y = \left(\frac{1}{2}a^2\right)\left(\frac{1}{2}b^2\right) c = \frac{1}{4}a^2b^2c \;,$$

which agrees with the volume integral of $\nabla \cdot \mathbf{F}$ above.

Example 4.7. A more typical example of the use of the Divergence Theorem is the following. Find the integral $\int_S \mathbf{A}.\mathrm{d}\mathbf{S}$ for $\mathbf{A} = (x, z, 0)$ and the surface S of a sphere of radius a.

Using the divergence theorem, the surface integral is equal to the volume integral $\int_V \nabla \cdot \mathbf{A}\, dV$ over the volume V interior to the sphere. But $\nabla \cdot \mathbf{A} = 1$, so the volume integral is $\int 1 \,\mathrm{d}V$ over the sphere, which is the volume of the sphere $= 4\pi a^3/3$.

Doing the surface integral $\int_S \mathbf{A} \cdot d\mathbf{S}$ directly is possible, but much more long-winded.

Example 4.8. Another good example is that from Example 4.5, where we evaluated a rather fiddly surface integral over 1/8th of a sphere. In that case, we were given $\mathbf{F} = z\mathbf{k}$; so $\nabla \cdot \mathbf{F} = 1$; and the Divergence theorem tells us that a volume integral of $\nabla \cdot \mathbf{F}$ is equal to the surface integral of $\mathbf{F} \cdot d\mathbf{S}$ over the *whole* surface bounding the volume. We may choose our volume as the interior of the 1/8 sphere, bounded by three planes $x = 0$, $y = 0$, $z = 0$ and the 1/8 sphere $x^2 + y^2 + z^2 = a^2$ with $x, y, z > 0$, then the volume integral of $\nabla \cdot \mathbf{F}$ is just $(1/8)$ (Volume of full sphere) $= \pi a^3/6$.

The surface integral is the sum of four parts: one part over the 1/8 surface of the sphere which we did before, plus three surface integrals over flat quarter-circles in each of the xy, xz and yz planes: those have *outward* unit normal vectors $-\mathbf{k}$, $-\mathbf{j}$, $-\mathbf{i}$ respectively since our volume is on the positive side of each plane. But $\mathbf{F} = z\mathbf{k}$, so for the second and third of those planes the dot product $\mathbf{F}.d\mathbf{S}$ is zero; and for the first plane, we are at $z = 0$ so $\mathbf{F} = 0$. Therefore, all three of the flat quarter-circles give us surface integrals of 0 ; so the surface integral of $\mathbf{F}.d\mathbf{S}$ over the 1/8 sphere is equal to the volume integral of $\nabla \cdot \mathbf{F}$, $= \pi a^3/6$, QED.

Exercise 4.5. State the Divergence Theorem. Evaluate both sides of the Divergence Theorem for the vector field $\mathbf{F} = xy^2z\,\mathbf{k}$ over a volume V which is the interior of the unit cube, i.e. the cube whose vertices are at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ and $(1, 1, 1)$. □

The Divergence Theorem equates two scalar values. However, one can derive from it vector identities. For example, we can obtain what is called the vector form of the theorem:

$$\int U \,\mathrm{d}\mathbf{S} = \int \nabla U \,\mathrm{d}V \qquad , \tag{4.9}$$

where U is a scalar field, and both sides of the above equation are vectors.

This is proved as follows: given the scalar field U, we choose any constant vector $\mathbf{a}$ and define a new vector field $\mathbf{F} = U\,\mathbf{a}$; next we apply the usual divergence theorem to $\mathbf{F}$, and the product rule Eq. 3.6 gives us $\nabla \cdot (U\mathbf{a}) = 0 + \mathbf{a} \cdot (\nabla U)$, so

$$\int \mathbf{a}U \cdot \,\mathrm{d}\mathbf{S} = \int \mathbf{a} \cdot (\nabla U)\,\mathrm{d}V \qquad .$$

Since $\mathbf{a}$ is a constant vector we can take it outside the integral signs, and finally choosing the cases $\mathbf{a} = \mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in turn, we prove Eq. 4.9.

4.5 Green's Theorem (in the plane)

(See Thomas 16.4: we take the statement he gives as Theorem 4, reworded. Note that the right side is a component of a curl.)

Theorem 4.2 *(Green's Theorem:) If $\mathscr{C}$ is a simple closed curve in the x-y plane, traversed counterclockwise, and M and N are suitably differentiable functions of x and y, then*

$$\int_{\mathscr{C}} (M\mathrm{d}x + N\mathrm{d}y) = \int\int_{\mathscr{R}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y,$$

where the area integral is over the region $\mathscr{R}$ enclosed by the curve $\mathscr{C}$.

Note that if asked to state the theorem you must state the nature of $\mathscr{C}$ ("simple closed") and the direction in which it is travelled.

Proof: The proof is an application of the Divergence Theorem, choosing a volume of height 1 in the z-direction above $\mathscr{R}$. (Or, if one proves Stokes's theorem first, of that theorem.) Take $\mathbf{F} = (N, -M, 0)$: then

$$\begin{aligned} \int (\nabla \cdot \mathbf{F}) \mathrm{d}V &= \int_{z=0}^{1} \int\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y \end{aligned}$$

on integrating over z from 0 to 1. On the top and bottom of the volume, $\mathrm{d}\mathbf{S}$ is in the $\pm\mathbf{k}$ direction so $\mathbf{F}.\mathrm{d}\mathbf{S} = 0$. On the rest of the surface we have

$$\int \mathbf{F}.\mathrm{d}\mathbf{S} = \int\int N\mathrm{d}S_x - M\mathrm{d}S_y$$

where $\mathrm{d}S_x$ is the component of $\mathrm{d}\mathbf{S}$ along the x-axis. Using $\mathrm{d}\mathbf{r}_{\mathscr{C}} = (\mathrm{d}x, \mathrm{d}y, 0)$ along $\mathscr{C}$ and $\mathrm{d}\mathbf{r}_z = (0, 0, \mathrm{d}z)$ in the z-direction, $\mathrm{d}\mathbf{S} = \mathrm{d}\mathbf{r}_{\mathscr{C}} \times \mathrm{d}\mathbf{r}_z$ gives $\mathrm{d}S_x = \mathrm{d}y\,\mathrm{d}z$ and $\mathrm{d}S_y = -\mathrm{d}x\,\mathrm{d}z$, so

$$\begin{aligned} \int \mathbf{F}.\mathrm{d}\mathbf{S} &= \int\int_S N\ \mathrm{d}y\,\mathrm{d}z + \int\int_S M\ \mathrm{d}x\,\mathrm{d}z \\ &= \int_C N\ \mathrm{d}y + \int_C M\ \mathrm{d}x. \end{aligned}$$

where the second line follows because the z integral runs from 0 to 1 and the integrand is independent of z; now we have proved the two sides of the theorem are equal.

(Thomas's Theorem 3 is the same with N replaced by M and M replaced by $-N$. This version makes the right side look like a two-dimensional divergence. Sometimes you may see these called Green's theorem (first form) and Green's Theorem (second form) etc.)

Example 4.9. Use Green's theorem to evaluate

$$\int \left(xy\ \mathrm{d}y - y^2\ \mathrm{d}x\right)$$

around the unit square: straight path segments from the origin to $(1,0)$ to $(1,1)$ to $(0,1)$ and back to the origin.

In this case, $M = -y^2$ and $N = xy$; hence

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = y + 2y = 3y$$

Thus the required integral is

$$\int_0^1 \int_0^1 3y\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 (3/2)\,\mathrm{d}x = 3/2.$$

4.5.1 Area within a curve

From Green's Theorem, we can get a surprising expression for the area A inside a closed curve C bounding a region S in a plane is

$$\mathbf{n}A = \tfrac{1}{2}\oint_C \mathbf{r}\times \mathrm{d}\mathbf{r}\ ,$$

where $\mathbf{n}$ is the unit normal to the plane. We can assume without loss of generality that the plane of the curve is the x,y plane. Then $\mathbf{n} = \mathbf{k}$ and $\mathbf{r}\times \mathrm{d}\mathbf{r} \propto \mathbf{k}$, so we only need the z component of the integral which is

$$\tfrac{1}{2}\oint_C (x\,dy - y\,dx).$$

By Green's theorem in the plane this equals

$$\frac{1}{2}\int_S 2\;dx\,dy = \int_S 1\;dx\,dy = \text{ Area inside } C.$$

This can be useful for example if we are given a curve in parametric form $(x,y) = (f(t), g(t))$ which contains a closed loop, and we want the area of the loop: since the curve has a closed loop, then there are two values of t_1, t_2 where the curve returns to the same point, and (as long as the curve does not cross itself between t_1, t_2), we can evaluate the enclosed area within that loop using the above formula as

$$A \quad = \quad \frac{1}{2}\int_{t_1}^{t_2} x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt}\;dt \tag{4.10}$$

A neat example of this is the case of the ellipse, $x = a\cos t, \quad y = b\sin t$; this clearly is a closed loop for $t_1 = 0,\ t_2 = 2\pi$, and we obtain the area as

$$A = \frac{1}{2}\int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)\;dt = \pi ab \qquad .$$

4.6 Stokes's Theorem

(See Thomas 16.7)
The other major theorem of similar character to the Divergence Theorem is Stokes's theorem which follows. (Because both are versions of the n-dimensional Stokes's theorem, we can prove Stokes's theorem from Green's and thence from the Divergence Theorem, which we do in section 4.9. It can also be proved directly.) We reword Thomas's version.

Theorem 4.3 *[Stokes's theorem]: If* $\mathbf{F}$ *is a (suitably differentiable) vector field, and* $\mathscr{C}$ *is a closed path bounding an oriented surface* $\mathscr{S}$*, then*

$$\int_{\mathscr{C}} \mathbf{F}.\mathrm{d}\mathbf{r} = \int_{\mathscr{S}} (\nabla\times\mathbf{F}).\mathbf{n}\;\mathrm{d}S \equiv \int_{\mathscr{S}} (\nabla\times\mathbf{F}).\mathrm{d}\mathbf{S}, \tag{4.11}$$

where $\mathscr{C}$ *is travelled counterclockwise with respect to the unit normal* $\mathbf{n}$ *of* $\mathscr{S}$ (i.e. counterclockwise as seen from the positive $\mathbf{n}$ side of $\mathscr{S}$).

Again note that if asked to state the theorem, you must state that $\mathscr{C}$ is closed, that it bounds $\mathscr{S}$, and that the directions of d$\mathbf{S}$ and d$\mathbf{r}$ are related as given above.

It is easy to show that Green's theorem is a planar version of this result.

Note that the result is the same for any surface $\mathscr{S}$ whose boundary is $\mathscr{C}$, so any two surfaces $\mathscr{S}_1$, $\mathscr{S}_2$ with the same bounding curve $\mathscr{C}$ give the same surface integral. (We will not give a formal proof of this, but in a nutshell it is because
$\nabla\cdot(\nabla\times\mathbf{F}) = 0$ from Eq. 3.12, then applying the Divergence theorem to the volume enclosed between the two surfaces). This can simplify integration a lot if the bounding curve lies in a plane, since we can replace a surface integral over a curved surface with that over the flat surface with the same boundary.

To emphasize the need for differentiability conditions, consider

$$\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}.$$

We can easily verify that $\nabla\times\mathbf{F} = \mathbf{0}$ (except on the z axis where it diverges). But we can also show that $\oint \mathbf{F}.\mathrm{d}\mathbf{r} \neq 0$ if we go around the z axis: for example going round a circle of radius a using a parametrization $(a\cos\theta, a\sin\theta)$we would have

$$\int \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \oint a^{-2}(-a\sin\theta\mathbf{i}+a\cos\theta\mathbf{j}).(-a\sin\theta\mathbf{i}+a\cos\theta\mathbf{j})\,\mathrm{d}\theta = \oint\,\mathrm{d}\theta = 2\pi.$$

This occurs because our closed curve has looped around the $z-$axis where there is infinite curl; if you do the Complex Variables module in Semester B, this is very similar to a contour integral around a pole.

Example 4.10. Use the surface integral in Stokes's theorem to calculate the circulation of the field $\mathbf{F}$

$$\mathbf{F} = x^2\mathbf{i}+2x\mathbf{j}+z^2\mathbf{k}$$

around the curve $\mathscr{C}$, where $\mathscr{C}$ is the ellipse $4x^2+y^2 = 4$ in the x-y plane, taken counterclockwise when viewed from $z > 0$.

In Stokes's Theorem, we can choose **any** surface that spans the curve $\mathscr{C}$. The easiest one in this case is just the planar surface $z = 0$ contained inside the ellipse (so we can use Green's theorem in fact). Thus $\mathbf{n}$ will be purely in the z-direction: $\mathbf{n} = \mathbf{k}$, and so we only need to calculate the z-component of $\nabla\times\mathbf{F}$:

$$(\nabla\times\mathbf{F}).\mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial(2x)}{\partial x} - \frac{\partial x^2}{\partial y} = 2.$$

Integrating this over the elliptical area is easy: the answer is just 2 times the area of the ellipse. The area of an ellipse is πab, where a is one semi-major axis length (in this case 1) and b is the other semi-major axis length (in this case 2). Hence the answer is 4π.

As in the case of the Divergence Theorem, we can give a vector form of Stokes's Theorem. Given a scalar field U, we let $\mathbf{F} = U\mathbf{a}$ for some constant vector $\mathbf{a}$. Then

$$\begin{aligned}\int_{\mathscr{C}} U\mathbf{a}\cdot\mathrm{d}\mathbf{r} &= \int_{\mathscr{S}}(\nabla\times(U\mathbf{a}))\cdot d\mathbf{S}\\ &= \int_{\mathscr{S}}((\nabla U)\times\mathbf{a})\cdot\mathrm{d}\mathbf{S}\\ &= \mathbf{a}.\int_{\mathscr{S}}\mathrm{d}\mathbf{S}\times(\nabla U).\end{aligned}$$

The first line is Stokes' theorem, the second follows from the rule Eq 3.8 for curl of a product, and the third from the rules for the scalar triple product. Now we can take the constant $\mathbf{a}$ outside the integral sign; then choosing **a** = **i**, **j** and **k** in turn, we derive the vector equation

$$\int_{\mathscr{C}} U \, \mathrm{d}\mathbf{r} = \int_{\mathscr{S}} \mathrm{d}\mathbf{S} \times (\nabla U) \qquad .$$

Exercise 4.6. State Stokes's theorem.
Evaluate both sides of the theorem for the vector field $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + y\mathbf{k}$ and the surface S of the hemisphere $x^2 + y^2 + z^2 = 4$ in $z \geq 0$, with normal in the positive z-direction. [You may find the expressions relating Cartesian and spherical polar coordinates useful.] □

Exercise 4.7. Use the surface integral in Stokes's theorem to calculate the circulation of the field **F**

$$\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$$

around the curve $\mathscr{C}$ where $\mathscr{C}$ is the circle $x^2 + y^2 = 9$ in the x-y plane, counterclockwise when viewed from $z > 0$. [Answer 9π.] □

We can use the Divergence and Stokes's theorems to derive other results including, later on, the forms of divergence and curl in curvilinear coordinates in Chapter 5. Those formulas could be found, more laboriously, by direct calculation from the Cartesian definitions by applying the chain rule. Another important application will be given next.

4.7 Conservative Fields and Scalar Potentials

(See Thomas 16.3)
Conservative vector fields play an important role in many applications. A vector field **F** is said to be a **conservative field** iff the value of the line integral $\int_P^Q \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ between endpoints P and Q depends only on the endpoints P and Q, and **not** on the path taken between them. An example of a vector field which is not conservative is the one in Example 4.2 – we explicitly found different answers for the same endpoints, depending on the path taken.

For a conservative vector field **F**, the integral $\int \mathbf{F}.\mathrm{d}\mathbf{r}$ around any **closed** path must be zero (because the value will be given by the trivial path which always stays at the given point). So if **F** is a force, for example, the net work in going round a path back to where one started is zero: energy is conserved, hence the name conservative (nothing to do with politics).

We first state and prove the important result that (subject to differentiability conditions) a vector field is conservative iff it is irrotational (or curl-free). In its statement, 'contractible' means we can continuously deform the region so it squashes to a point. (A torus, for example, is not contractible.)

Theorem 4.4 *In a contractible region,*

$$\nabla \times \mathbf{F} = \mathbf{0} \quad \Longleftrightarrow \quad \exists \text{a scalar field } \phi(\mathbf{r}) \text{ such that } \mathbf{F} = \nabla\phi. \tag{4.12}$$

Note: Such a ϕ is called a **(scalar) potential** for **F**. The theorem says a vector field is conservative iff it has a scalar potential.

Proof:

($\Leftarrow$): This was done at the end of Chapter 3, where we proved the identity $\nabla \times (\nabla\phi) = \mathbf{0}$ for any ϕ, subject to the partial derivatives being well-behaved. Thus if $\mathbf{F} = \nabla\phi$ then $\nabla \times \mathbf{F} = \mathbf{0}$.

($\Rightarrow$): Given $\nabla \times \mathbf{F} = \mathbf{0}$, we proceed by defining the scalar field $\phi(\mathbf{r})$ by

$$\phi(\mathbf{r}) = \int_{\mathbf{a}}^{\mathbf{r}} \mathbf{F}.\mathrm{d}\mathbf{r}, \tag{4.13}$$

where $\mathbf{a}$ is an arbitrary but fixed point; note the line integral has a scalar answer, so ϕ is a scalar field. We will soon show that $\nabla\phi = \mathbf{F}$ as required. First though, since we have not defined the path to be taken from $\mathbf{a}$ to $\mathbf{r}$, we must show that the integral is independent of the path taken, i.e. that the ϕ defined above is well-defined.

Suppose that $\mathscr{C}_1$ and $\mathscr{C}_2$ are two different curves from $\mathbf{a}$ to $\mathbf{r}$. We need to show that

$$\int_{\mathscr{C}_1} \mathbf{F}.\mathrm{d}\mathbf{r} = \int_{\mathscr{C}_2} \mathbf{F}.\mathrm{d}\mathbf{r}.$$

To prove this, let $\mathscr{C}$ be the closed curve formed by following $\mathscr{C}_1$ from $\mathbf{a}$ to $\mathbf{r}$ and then taking $\mathscr{C}_2$ **backwards** to get from $\mathbf{r}$ back to $\mathbf{a}$. Let $\mathscr{S}$ be a surface whose boundary is $\mathscr{C}$. Then:

$$\begin{aligned} \int_{\mathscr{C}_1} \mathbf{F}.\mathrm{d}\mathbf{r} - \int_{\mathscr{C}_2} \mathbf{F}.\mathrm{d}\mathbf{r} &= \int_{\mathscr{C}} \mathbf{F}.\mathrm{d}\mathbf{r} \\ &= \int_{\mathscr{S}} (\nabla \times \mathbf{F}).\mathrm{d}\mathbf{S} \\ &= 0 \end{aligned}$$

The first line is because following $\mathscr{C}_\in$ backwards gives us a minus sign in the line integral; the second line is Stokes' theorem for the closed curve $\mathscr{C}$. Hence, the value of ϕ only depends on $\mathbf{r}$, but **not** on the path taken from $\mathbf{a}$ to $\mathbf{r}$, and so $\phi(\mathbf{r})$ **is** well-defined. [Note: Thomas gives a direct proof of the path-independence property for $\mathbf{F} = \nabla V$.]

Next we need to show $\nabla\phi = \mathbf{F}$ as we wanted: we consider a small change $\delta\mathbf{r}$, and we get a small change $\delta\phi$,

$$\delta\phi \equiv \phi(\mathbf{r} + \delta\mathbf{r}) - \phi(\mathbf{r}) = \int_{\mathbf{r}}^{\mathbf{r}+\delta\mathbf{r}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \approx \mathbf{F}(\mathbf{r}) \cdot \delta\mathbf{r} \;,$$

and this is true for any (infinitesimal) vector $\delta\mathbf{r}$. But by definition of $\nabla\phi$ in Chapter 1, $\delta\phi = (\nabla\phi) \cdot \delta\mathbf{r}$. Hence

$$\nabla\phi \cdot \delta\mathbf{r} = \mathbf{F} \cdot \delta\mathbf{r} \;.$$

But this is true for *all* $\delta\mathbf{r}$, so $\nabla\phi = \mathbf{F}$, as we wanted to show. Q.E.D.

Once we have done this, we easily get the line integral $\int \mathbf{F} \cdot d\mathbf{r}$ between any two points, say $\mathbf{r}_1$ to $\mathbf{r}_2$: choose a path from $\mathbf{r_1}$ back to $\mathbf{a}$, and then from $\mathbf{a}$ to $\mathbf{r_2}$; since taking a line integral backwards gives us a minus sign in the result (as for swapping upper/lower limits in a 1D integral), we get

$$\int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{F} \cdot d\mathbf{r} = \phi(\mathbf{r_2}) - \phi(\mathbf{r_1})$$

Also note that we can add a constant to ϕ without changing $\nabla\phi$; adding a constant is essentially equivalent to changing our choice of fixed point $\mathbf{a}$ in eq. 4.13, since $\phi(\mathbf{a}) = 0$ from the original definition.

In the case where $\mathbf{F}$ is a force, it is usual to define $\phi(\mathbf{r}) = -\int_{\mathbf{a}}^{\mathbf{r}} \mathbf{F}.\mathrm{d}\mathbf{r}$ with an extra (arbitrary) minus sign compared to (4.13); then we get $\mathbf{F} = -\nabla\phi$, and ϕ can then be identified with the potential energy, which decreases when a body moves in the direction of the force "down", and increases in the opposite direction "up". Note again that the value of ϕ is only fixed up to an additive constant, which depends on the choice of reference point $\mathbf{a}$.

Warning: There is a possible snag with notation here: it is very common for historical reasons to use the symbol ϕ (the Greek letter "phi") for a scalar potential, or sometimes V by analogy with Voltage in electrostatics. That ϕ is obviously **not** related to the coordinate angle ϕ which will appear later in spherical polar coordinates; or also V can possibly get confused with volume. Sometimes the symbols Φ (uppercase phi) or φ (curly phi) are used for the potential, but this still looks quite similar.

Unfortunately, this somewhat confusing notation is heavily used in many textbooks and old exam questions, so it can't be escaped and you just have to be aware of it. In most cases it is reasonably obvious from the context which is which.

Example 4.11. Show that $\mathbf{F} = (z, z, x+y)$ satisfies $\nabla \times \mathbf{F} = 0$, and find a scalar field ϕ such that $\mathbf{F} = \nabla\phi$.

[Note that in answering questions of this sort, where you have to find ϕ, you might as well do that first since $\mathbf{F} = \nabla\phi$ immediately implies $\nabla \times \mathbf{F} = 0$.]

A simple way to do these problems is by direct evaluation of the line integral (4.13), taking as the curve $\mathscr{C}$ as the straight line from the origin (so we are taking $\mathbf{a}$ to be the origin) to the desired point, (X, Y, Z) say. The line is $\mathbf{r} = t(X, Y, Z)$, $0 \le t \le 1$, so $\mathrm{d}\mathbf{r} = (X, Y, Z)\mathrm{d}t$, while for this example, on that line $\mathbf{F} = (Zt, Zt, Xt+Yt)$. Thus the integral is

$$\int_{\mathscr{C}} \mathbf{F}.\mathrm{d}\mathbf{r} = \int_0^1 [XZt + YZt + (Xt+Yt)Z]\mathrm{d}t = (2XZ+2YZ)\int_0^1 t\,\mathrm{d}t = (2XZ+2YZ)[t^2]_0^1 = XZ+YZ.$$

Hence for a general point we have $\phi = xz + yz$. We can also add any constant to ϕ (since it will disappear in $\nabla\phi$): this expresses the freedom of choice of the $\mathbf{a}$ in (4.13). [In physical uses of scalar potentials, the reference point is often taken to be at infinity.]

An alternative method is as follows: it is included to emphasize some useful points about integrating sets of partial differential equations (i.e. differential equations with partial derivatives).

We want

$$(z, z, x+y) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right). \tag{4.14}$$

Equating the first components and integrating with respect to x gives

$$z = \frac{\partial\phi}{\partial x} = z \Rightarrow \phi = xz + f(y,z) \tag{4.15}$$

where f is an (as yet) arbitrary function of y and z. Note that f is a 'constant of integration' as far as differentiation with respect to x is concerned: when integrating partial derivatives we have to replace simple constants by functions of those variables not yet taken into account. The second components give

$$z = \frac{\partial\phi}{\partial y} \text{ from (4.14)} = \frac{\partial f}{\partial y} \text{ from (4.15)}.$$

Hence

$$\frac{\partial f}{\partial y} = z \Rightarrow f(y,z) = yz + g(z)\ .$$

No x appears in g since we already know that f does not depend on x. So, substituting this in (4.15),

$$\phi = xz + yz + g(z) \tag{4.16}$$

(g arbitrary as yet). Finally, the third components similarly give

$$x+y = \frac{\partial\phi}{\partial z} \text{ from (4.14)} = x+y+\frac{\mathrm{d}g}{\mathrm{d}z} \text{ from (4.16)}.$$

Hence g has a zero derivative, i.e. is constant and there is a ϕ given by

$$\phi = xz + yz + \text{const.}$$

(We could drop the constant here as without it ϕ would still fulfil the conditions of the problem.) Hence $\nabla \times \mathbf{F} = \mathbf{0}$.

Example 4.12. The gravitational force on a ball of mass m is $\mathbf{F} = (0,0,-mg)$. If the gravitational acceleration g can be assumed to be constant (which is an excellent approximation for everyday life: $g \simeq 9.8\text{ms}^{-2}$) then $\mathbf{F} = -\nabla\phi$ where $\phi = mgz+$const., z being measured, say, from the surface of the Earth. (We can measure z from wherever we wish, since a change of origin just changes the arbitrary constant in ϕ). In this case ϕ is the **gravitational potential energy**.

Exercise 4.8. Show that $\mathbf{F} = (yz, zx, xy)$ is conservative and find a suitable potential ϕ such that $\mathbf{F} = \nabla\phi$. [Answer: $\phi = xyz+$const.] □

Exercise 4.9. For each of the following fields $\mathbf{F}$, evaluate $\nabla \times \mathbf{F}$ and either find the general solution ϕ satisfying $\mathbf{F} = \nabla\phi$ everywhere, or show that no such ϕ exists:

(a) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + 2z\mathbf{k}$

(b) $\mathbf{F} = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$

(c) $\mathbf{F} = 3z^2\mathbf{i} + 3y^2\mathbf{j} + 6xz\mathbf{k}$

(d) $\mathbf{F} = yz\mathbf{j} - xy\mathbf{k}$.

□

The rest of this chapter will not be lectured and is not examinable. It is included for reference, for completeness, and to give intellectual respectability by proving the main theorems.

4.8 Vector Potentials

(Note: this is not on the syllabus. It is included for completeness, for the sake of those who take later courses where it is used.)

We have seen that, if $\nabla \times \mathbf{F} = \mathbf{0}$, then there exists a scalar potential ϕ such that $\mathbf{F} = \nabla\phi$. There is a similar result if $\nabla \cdot \mathbf{F} = 0$ instead:

Theorem 4.5 *In a contractible domain,*

$$\nabla \cdot \mathbf{F} = 0 \qquad \Longleftrightarrow \qquad \exists \mathbf{A}(\mathbf{r}) \text{ such that } \mathbf{F} = \nabla \times \mathbf{A}.$$

In the ($\Leftarrow$) direction, this is the identity discussed before. The proof in the other direction consists of writing down suitable integrals, in a way analogous to the proof of (4.12), and is messy so we omit it.

The function $\mathbf{A}$ is called a **vector potential**. Note that one can always add an arbitrary function of the form $\nabla\phi$ to $\mathbf{A}$ and get another perfectly good vector potential for $\mathbf{F}$, because $\nabla \times (\nabla\phi)$ is zero for any ϕ, and so

$$\nabla \times (\mathbf{A} + \nabla\phi) = \nabla \times \mathbf{A} + \nabla \times (\nabla\phi) = \mathbf{F} + \mathbf{0} = \mathbf{F}.$$

In physical contexts this is referred to as a gauge transformation, and provides the basic example whose generalization gives all the modern gauge field theories of physics, the basis of our understanding of all microsopic physical processes.

Example 4.13. Any magnetic field $\mathbf{B}$ satisfies $\nabla\cdot\mathbf{B}=0$. So, for example, consider a constant magnetic field $\mathbf{B}=(0,0,B_0)$ in the $z-$direction. A suitable vector potential $\mathbf{A}$ in this case is

$$\left(-\frac{1}{2}B_0y,\frac{1}{2}B_0x,0\right),$$

since

$$\begin{aligned}\nabla\times\mathbf{A} &= \left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z},\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x},\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\\ &= \left(0-0,0-0,\frac{1}{2}B_0-(-\frac{1}{2}B_0)\right)\\ &= \mathbf{B}.\end{aligned}$$

4.9 Derivations of the main theorems

(See Thomas 16.7 and 16.8)
[This section is not examinable]

We now return to the proofs of the Divergence and Stokes's Theorems.

Consider first the "proof" of the Divergence Theorem using rectangular boxes. Take a box $[x_1,x_2]\times[y_1,y_2]\times[z_1,z_2]$. Then for a vector $\mathbf{A}=A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$,

$$\begin{aligned}\int(\nabla\cdot\mathbf{A})\mathrm{d}V &= \int\int\int\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &= \int\int[A_1]_{x_1}^{x_2}\mathrm{d}y\,\mathrm{d}z+\int\int[A_2]_{y_1}^{y_2}\mathrm{d}x\,\mathrm{d}z+\int\int[A_3]_{z_1}^{z_2}\mathrm{d}x\,\mathrm{d}y\\ &= \int\int_{\text{front}}A_1\mathrm{d}y\,\mathrm{d}z-\int\int_{\text{back}}A_1\mathrm{d}y\,\mathrm{d}z+\int\int_{\text{right end}}A_2\mathrm{d}x\,\mathrm{d}z-\int\int_{\text{left end}}A_2\mathrm{d}x\,\mathrm{d}z \qquad (4.17)\\ &\quad+\int\int_{\text{top}}A_3\mathrm{d}x\,\mathrm{d}y-\int\int_{\text{bottom}}A_3\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

On the front of the box (i.e. the surface $x=x_2$) $\mathrm{d}\mathbf{S}=\mathbf{i}\mathrm{d}y\mathrm{d}z$ while on the back ($x=x_1$) $\mathrm{d}\mathbf{S}=-\mathbf{i}\mathrm{d}y\mathrm{d}z$ so the first two terms in (4.17) are $\int\mathbf{A}.\mathrm{d}\mathbf{S}$ for the front and back. Similarly for the remaining terms.

One can complete a "proof" by decomposing a volume into such boxes and adding the results, noting that the surface integrals on a face common to two boxes will cancel one another. This overlooks the difficulty of proving that the surface integral for all the boxes gives a correct limit for the smooth surface (for the volume integral this just follows from the definition of such integrals).

Instead we can work towards a correct proof by first noting that the terms match up in the sense that

$$\int\int\int_{\mathscr{D}}\frac{\partial A_3}{\partial z}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int\int_{\mathscr{S}}A_3(\mathrm{d}S)_z \qquad (4.18)$$

for the box. (What we thus really do is prove the theorem for $\mathbf{F}=A_3\mathbf{k}$ and then add together three such results.)

We now have to cope with some technical points
1. We must be able to integrate the derivatives of **A** once. A sufficient condition is that all first derivatives of **A** are piecewise continuous. If the derivatives have discontinuities we have to do the proof for each smooth piece separately and then add the results.
2. That first point implies **A** itself must be piecewise continuous.
3. We require the surface to be bounded (so we have a finite area) and closed (so we have a finite volume).
4. We must to be able to integrate $\int \mathbf{A}.\mathrm{d}\mathbf{S}$. So we want to be able to assign coordinates on pieces of the surface S, say (u, v), in such a way that $(\mathbf{e}_u \times \mathbf{e}_v)\mathrm{d}u\,\mathrm{d}v$ can be defined and calculated, i.e. we want the map $\mathbb{R}^2 \to \mathbb{R}^3 : (u, v) \to (x(u, v), y(u, v), z(u, v))$ to be (piecewise) sufficiently differentiable.

These assumptions ensure we can break D up into convex pieces. 'Convex' means that any line cuts the surface at most twice. So now we have the form

Theorem 4.6 *If $\mathscr{S}$ is a bounded closed piecewise smooth orientable surface enclosing a volume $\mathscr{D}$, and if* **F** *is a vector field all of whose first derivatives are continuous, then*

$$\int_{\mathscr{D}} \nabla \cdot \mathbf{F}\,\mathrm{d}V = \int_{\mathscr{S}} \mathbf{F}.\mathbf{n}\,\mathrm{d}S = \int_{\mathscr{S}} \mathbf{F}.\mathrm{d}\mathbf{S},$$

where **n** *is the normal outward-pointing from $\mathscr{D}$.*

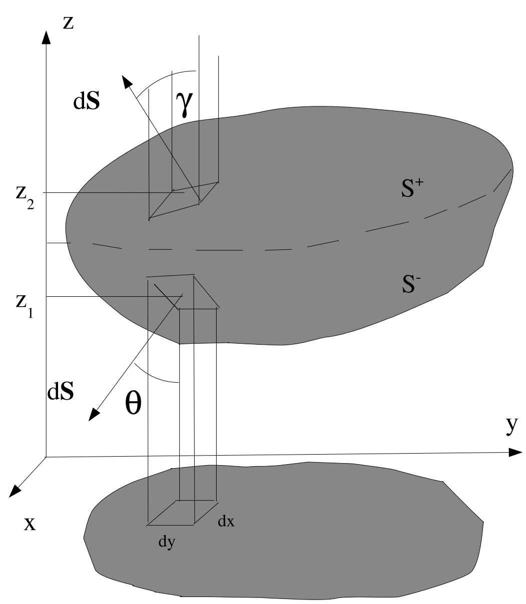

Figure 4.3: Convex surface used in the proof of the Divergence Theorem

Proof: [This proof is more-or-less identical, with slight changes in notation, with the one given by Thomas.] We break $\mathscr{D}$ into convex pieces and first prove the result for a single convex piece (which we call $\mathscr{D}_1$). In fact we need only prove (4.18). Consider lines parallel to the z-axis. Those which meet $\mathscr{D}_1$ either meet it twice or touch it on a closed curve. Divide the surface into $\mathscr{S}^+$ and $\mathscr{S}^-$, the upper and lower halves (i.e. $\mathscr{S}^-$ is where the lines parallel to the z-axis first meet $\mathscr{S}$: see Figure 4.3). Then, just using the fundamental theorem of calculus,

$$\int\int\int_{\mathscr{D}_1} \frac{\partial A_3}{\partial z}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\int_{\mathscr{S}^+} A_3(x, y, z_2)\mathrm{d}x\,\mathrm{d}y - \int\int_{\mathscr{S}^-} A_3(x, y, z_1)\mathrm{d}x\,\mathrm{d}y$$

On $\mathscr{S}^+$, $(A_3\mathbf{k}).\mathrm{d}\mathbf{S} = A_3|\mathrm{d}\mathbf{S}|\cos\gamma = A_3\mathrm{d}x\,\mathrm{d}y$ and similarly on $\mathscr{S}^-$. Hence we have shown that

$$\int\int\int_{\mathscr{D}_1} \frac{\partial A_3}{\partial z}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\int_{\mathscr{S}} (A_3\mathbf{k}).\mathrm{d}\mathbf{S}.$$

and adding similar results for A_1 and A_2 we get the Divergence Theorem for $\mathscr{D}_1$. When we re-combine the convex pieces, the surfaces where they join appear twice in the surface integrals, once with each of the two possible signs for the normal, so these parts cancel one another and only the integral over the bounding surface remains. Q.E.D.

We showed above that the Divergence Theorem implies Green's theorem. We only have Stokes's theorem left to prove. The conditions are arrived at by similar considerations to those for the Divergence Theorem.

Theorem 4.7 *For any piecewise smooth surface $\mathscr{S}$ bounded by a piecewise smooth curve $\mathscr{C}$ on which $\nabla\times\mathbf{F}$ is piecewise continuous,*

$$\int_{\mathscr{S}} \nabla\times\mathbf{F}.\mathrm{d}\mathbf{S} = \oint_{\mathscr{C}} \mathbf{F}.\mathrm{d}\mathbf{r},$$

where the integral round $\mathscr{C}$ is taken in the direction which is counter-clockwise as seen from the side of $\mathscr{S}$ pointed to by $\mathrm{d}\mathbf{S}$.

Proof: The conditions imply that the surface can be decomposed in pieces which project to regions in one of the planes of Cartesian coordinates; without loss of generality say the (x, y) plane. We prove the result for one such region. Suppose we have coordinates (u, v) on this region. We also consider only the terms involving P where $\mathbf{F} = (P, Q, R)$ (i.e. we prove the result for $\mathbf{F} = P\mathbf{i}$ first).

$$\begin{aligned}
\oint_{\mathscr{C}} P\mathrm{d}x &= \oint_{\mathscr{C}} P\left(\frac{\partial x}{\partial u}\mathrm{d}u + \frac{\partial x}{\partial v}\mathrm{d}v\right) \\
&= \int\int\left[-\frac{\partial}{\partial v}\left(P\left(\frac{\partial x}{\partial u}\right)\right) + \frac{\partial}{\partial u}\left(P\left(\frac{\partial x}{\partial v}\right)\right)\right]\mathrm{d}u\,\mathrm{d}v \ \text{ by Green's theorem} \\
&= \int\int\left(\frac{\partial P}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial P}{\partial v}\frac{\partial x}{\partial u}\right)\mathrm{d}u\,\mathrm{d}v \\
&= \int\int\left(\left(\frac{\partial P}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial v} - \left(\frac{\partial P}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial u}\right)\mathrm{d}u\,\mathrm{d}v \\
&\qquad \text{using the Chain Rule} \\
&= \int\int\frac{\partial P}{\partial y}\left(\frac{\partial y}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial x}{\partial u}\right)\mathrm{d}u\,\mathrm{d}v + \int\int\frac{\partial P}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right)\mathrm{d}u\,\mathrm{d}v
\end{aligned}$$

and taking the cross product of

$$\mathrm{d}\mathbf{r}_u = \left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\right)\mathrm{d}u,$$
$$\mathrm{d}\mathbf{r}_v = \left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\right)\mathrm{d}v,$$

easily shows that the double integrals give

$$\int\int\left(-\frac{\partial P}{\partial y}(\mathrm{d}\mathbf{S})_z + \frac{\partial P}{\partial z}(\mathrm{d}\mathbf{S})_y\right)$$

which is the part of $\nabla\times\mathbf{F}.\mathrm{d}\mathbf{S}$ involving P. To complete the proof we add the parts with Q and R and add together the results from the pieces into which a general $\mathscr{S}$ has to be split. Q.E.D.