Chapter 3

Vector differentiation, the ∇ operator, grad, div and curl.

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Syllabus topics covered:
1. Vector fields
2. Grad, div and curl operators in Cartesian coordinates. Grad, div, and curl of products etc.

Here we cover differentiation of vectors. Note that this differs from the gradient introduced in Chapter 1, where we obtained a vector by differentiating a scalar field.

3.1 Vector functions of one or more variables

(See Thomas 13.1)
In many physical contexts one is interested in vectors that vary with position or time. For example, the position of a point can be described by a vector $\mathbf{r}$. Thus, if we consider a moving particle, its position can be described as a function of time t by the vector $\mathbf{r}(t)$, and its rate of change with respect to t will be the velocity (which has magnitude and direction, i.e. is a vector: its magnitude is the speed). The position vector is then a function of one variable.

Another context is where we have a vector defined at each point, say $\mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y, z)$ and a curve with a parameter u, say, so its points are $(x(u), y(u), z(u))$. Then we can define a vector function of u, $\mathbf{F}(u) = \mathbf{F}((x(u), y(u), z(u))$. We can deal with this and the moving particle case as follows.

A vector function of a scalar u, $\mathbf{F}(u)$, can be defined by specifying its components as functions of u:

$$\mathbf{F}(u) = (f_1(u), f_2(u), f_3(u)).$$

The derivative $\mathrm{d}\mathbf{F}/\mathrm{d}u$ of $\mathbf{F}$ with respect to u is then:

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}u} = \left(\frac{\mathrm{d}f_1}{\mathrm{d}u}, \frac{\mathrm{d}f_2}{\mathrm{d}u}, \frac{\mathrm{d}f_3}{\mathrm{d}u}\right).$$

This simply goes back to the fundamental definition of a derivative:

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}u} = \lim_{\delta u \to 0} \frac{\mathbf{F}(u+\delta u) - \mathbf{F}(u)}{\delta u}.$$

Clearly one can compute higher derivatives, such as $\mathrm{d}^2\mathbf{F}/\mathrm{d}u^2$, by differentiating the components of $\mathbf{F}$ the required number of times.

Example 3.1. If $\mathbf{r}(t)$ is the position vector of a particle, as a function of time t, then $\mathrm{d}\mathbf{r}/\mathrm{d}t$ is the velocity $\mathbf{v}$ of the particle. Also $\mathrm{d}\mathbf{v}/\mathrm{d}t \equiv \mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$ is the particle's acceleration.

Example 3.2. The continuous parameter t can take all real values. Write down the derivatives $\mathrm{d}\mathbf{r}/\mathrm{d}t$ and $\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$ for the vector $\mathbf{r} = (\sin t)\mathbf{i} + t\mathbf{j}$. Also, sketch the curve whose parametric equation is $\mathbf{r} = \mathbf{r}(t)$.

The first and second derivatives are

$$\begin{aligned} \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= (\cos t)\mathbf{i} + \mathbf{j}, \\ \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} &= (-\sin t)\mathbf{i}. \end{aligned}$$

The sketch is shown in Fig. 3.1.

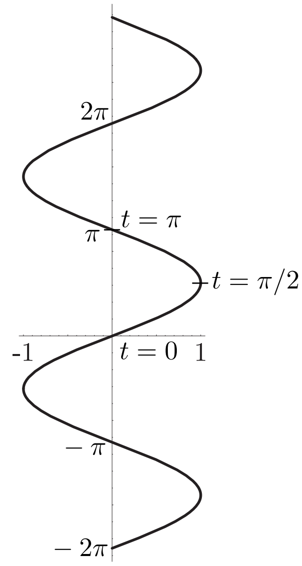

Figure 3.1: Sketch of the curve defined parametrically by $\mathbf{r} = (\sin t)\mathbf{i} + t\mathbf{j}$

It is easy to prove, by writing out the components and collecting terms, that if $\mathbf{F}$ and $\mathbf{G}$ are vector functions of u, then

$$\frac{\mathrm{d}(\mathbf{F}.\mathbf{G})}{\mathrm{d}u} = \mathbf{F}.\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}u} + \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}u}.\mathbf{G}.$$

Proof:

$$\begin{aligned} \frac{\mathrm{d}(\mathbf{F}.\mathbf{G})}{\mathrm{d}u} &= \frac{\mathrm{d}}{\mathrm{d}u}(f_1g_1 + f_2g_2 + f_3g_3) \\ &= f_1\frac{\mathrm{d}g_1}{\mathrm{d}u} + f_2\frac{\mathrm{d}g_2}{\mathrm{d}u} + f_3\frac{\mathrm{d}g_3}{\mathrm{d}u} + \frac{\mathrm{d}f_1}{\mathrm{d}u}g_1 + \frac{\mathrm{d}f_2}{\mathrm{d}u}g_2 + \frac{\mathrm{d}f_3}{\mathrm{d}u}g_3 \\ &= \mathbf{F}.\frac{\mathrm{d}\mathbf{G}}{\mathrm{d}u} + \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}u}.\mathbf{G}. \quad \text{Q.E.D.} \end{aligned}$$

It's also straightforward to show that cross products work the same way. **Exercise 3.1.** Sketch the curves whose parametric equations are

(a) $\mathbf{r} = (3\sin\pi t)\mathbf{i} + (2\cos\pi t)\mathbf{j}$

(b) $\mathbf{r} = (\cos\pi t)\mathbf{j}$

(c) $\mathbf{r} = t\mathbf{i} + t^2\mathbf{k}$

$(-\infty \le t \le \infty)$, and write down the derivatives $\mathrm{d}\mathbf{r}/\mathrm{d}t$ and $\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$ where they are defined. □

If $\mathbf{F}$ is a vector function of more than one variable, say $\mathbf{F} = \mathbf{F}(u, v)$, then it is straightforward to define its partial derivatives with respect to u or v, in terms of partial derivatives of its components. Thus, for example, if $\mathbf{F} = (f_1(u, v), f_2(u, v), f_3(u, v))$, then

$$\frac{\partial \mathbf{F}}{\partial u} = \left(\frac{\partial f_1}{\partial u}, \frac{\partial f_2}{\partial u}, \frac{\partial f_3}{\partial u}\right).$$

We have already met an example of this for the surface $\mathbf{r} = \mathbf{r}(u,v)$ in Chapter 2.

3.2 Vector Fields

(See Thomas 16.2)
For the rest of this course, we shall be concerned mostly with vectors and scalars which depend on position in three-dimensional space, i.e. which are functions of three variables x, y, z. We have already met a function $f(x,y,z)$ where f is one number (a scalar); from here on, this will be called a **scalar field**, where the word "field" means that it is a function of (x,y,z), and the "scalar" means the function value at each point is a scalar.

(**Note** : Sometimes things may depend also on a fourth variable, such as time t, or we may only be interested in their values on a particular path $\mathbf{r}(s)$ where s is a parameter; but this doesn't change the key results.)

A vector depending on position in 3-D space is said to constitute a **vector field**. We write a vector $\mathbf{F}$ that varies with position as

$$\mathbf{F} = \mathbf{F}(x,y,z) \equiv \mathbf{F}(\mathbf{r})$$

An example is shown in Figure 3.2. In order to actually specify a vector field $\mathbf{F}$, we need to write it out in terms of its components, each depending on position, so

$$\begin{aligned}\mathbf{F} &= (F_1(x,y,z),\ F_2(x,y,z),\ F_3(x,y,z)) \\ &= F_1(x,y,z)\,\mathbf{i} + F_2(x,y,z)\,\mathbf{j} + F_3(x,y,z)\,\mathbf{k} \quad .\end{aligned}$$

Clearly this is rather cumbersome so we'll often write $\mathbf{F}(\mathbf{r})$ or just $\mathbf{F}$; but remember to actually calculate things you'll often need to write it out in full.

We have already met one example of a vector field: given a scalar field U, we have defined the gradient as

$$\nabla U = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k} \quad .$$

Here ∇U is itself a vector, and it (usually) depends on position, so it is actually a vector field.

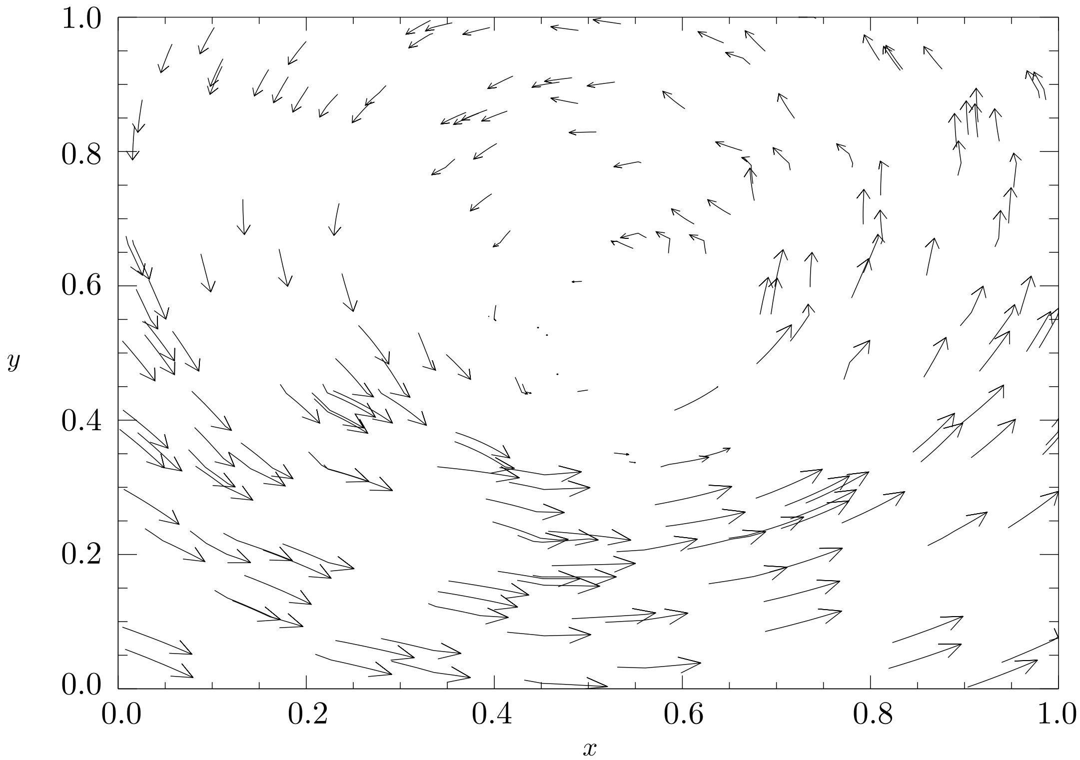

Figure 3.2: Example of a flow. In this case the speed and direction at each point is a function of the position (x,y)

A physical example of a vector field is the velocity in a flowing fluid (e.g. the water in the oceans, moving because of currents and tides; or the air in the atmosphere, moving because of winds). The velocity at any point in the fluid is a vector quantity – it has magnitude and direction. If we attach a velocity vector to each point of the flowing fluid, we have a vector field defined in the region occupied by the fluid.

Another physical example is a magnetic field; now things are not necessarily moving with time, but the magnetic field has a direction and a strength at each point in space; so at each point in space we have a vector; and this vector (in general) varies with position so it is a vector field.

We can add vector fields and multiply them by a constant in the obvious way, so if **F** and **G** are two vector fields then $\mathbf{F}+\mathbf{G}$ is also a vector field, and if λ is a constant then $\lambda\mathbf{F}$ is also a vector field.

Given a vector field, we could of course now differentiate the vector field with respect to each of the coordinates (x,y,z) in turn, in the manner described in the previous section; this gives us a total of 9 partial derivatives

$$\frac{\partial F_1}{\partial x},\ \frac{\partial F_2}{\partial x},\ldots,\frac{\partial F_1}{\partial y},\ldots,\frac{\partial F_3}{\partial z}$$

(In this course, we will be assuming that **F** is a smoothly-varying function of position, so all these derivatives exist at all points of interest **r**, except possibly for one or two singular points).

Note: the set of all 9 derivatives of a component by a coordinate forms a quantity of a new kind, called a **tensor**. These are used in fluid dynamics, solid mechanics and relativity, for example. However, in this course we will **not** deal with tensors, we will restrict ourselves to forming scalar and vector quantities from these 9 derivatives. To do this, it will turn out that we have to take certain special combinations which are "well behaved" if we rotate the x,y,z axes; these will turn out to be forming the dot and cross products of ∇

with **F** , where

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

is the operator called "**del**" which we met previously in forming the gradient of a scalar. Note again that ∇ is not a true vector (because on its own we can't define its length or direction), but it is a vector differential operator.

3.3 The Divergence of a vector field

(See Thomas 16.8)
Suppose $\mathbf{F}(x,y,z) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a vector field. The divergence of **F**, written div **F** or $\nabla \cdot \mathbf{F}$ is defined to be

$$\nabla \cdot \mathbf{F} \equiv \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{3.1}$$

Here div **F** is a scalar (there are no $\mathbf{i},\mathbf{j},\mathbf{k}$'s in the result) and generally depends on position, so it is a scalar field.

We can also get the above result if we write out ∇ and **F** in components,

$$\nabla \cdot \mathbf{F} = (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}) \cdot (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})$$

and write out all 9 terms then use the properties $\mathbf{i}.\mathbf{i} = 1$, $\mathbf{i}.\mathbf{j} = 0$, etc.

Note that, given a scalar field f, we found a vector field ∇f. Here, given a vector field **F**, we have produced a scalar field $\nabla \cdot \mathbf{F}$.

We can also write $\nabla \cdot \mathbf{F}$ as div **F**. These notations are completely interchangeable.

It is easy to show, by direct calculation, that div behaves as expected for addition and multiplication by a constant λ, i.e.

$$\nabla \cdot (\mathbf{F} + \mathbf{G}) = (\nabla \cdot \mathbf{F}) + (\nabla \cdot \mathbf{G}) \qquad ,$$
$$\nabla \cdot (\lambda \mathbf{F}) = \lambda (\nabla \cdot \mathbf{F})$$

The geometrical meaning of the divergence is as follows: consider a point **r** and consider a small closed surface surrounding that point: if the divergence div **F** is positive at **r**, then on average the vector field **F** is pointing "away" from the point and out of the surface. If the divergence is negative, then on balance **F** is pointing towards the point and into the surface. (See Fig. 3.3.) This idea will be made precise when we come to the Divergence Theorem in the next Chapter.

A vector field **F** for which $\nabla \cdot \mathbf{F} = 0$ everywhere is called **divergence-free** or **solenoidal**. The reason for the name **solenoidal** is historical: that a solenoid is a coiled wire that produces a magnetic field, and a magnetic field **B** is an example of a field that has $\nabla \cdot \mathbf{B} = 0$ everywhere (this is an observational fact, and arises because magnetic monopoles have never been found in many searches).

Example 3.3. If $\mathbf{F} = 3xy^2\mathbf{i} + e^z\mathbf{j} + xy\sin z\mathbf{k}$, calculate $\nabla \cdot \mathbf{F}$.

$$\nabla \cdot \mathbf{F} = \frac{\partial(3xy^2)}{\partial x} + \frac{\partial e^z}{\partial y} + \frac{\partial(xy\sin z)}{\partial z} = 3y^2 + xy\cos z.$$

Exercise 3.2. If $\mathbf{F} = (y-x)\mathbf{i} + (z-y)\mathbf{j} + (x-z)\mathbf{k}$, calculate $\nabla \cdot \mathbf{F}$. [Answer: -3] □

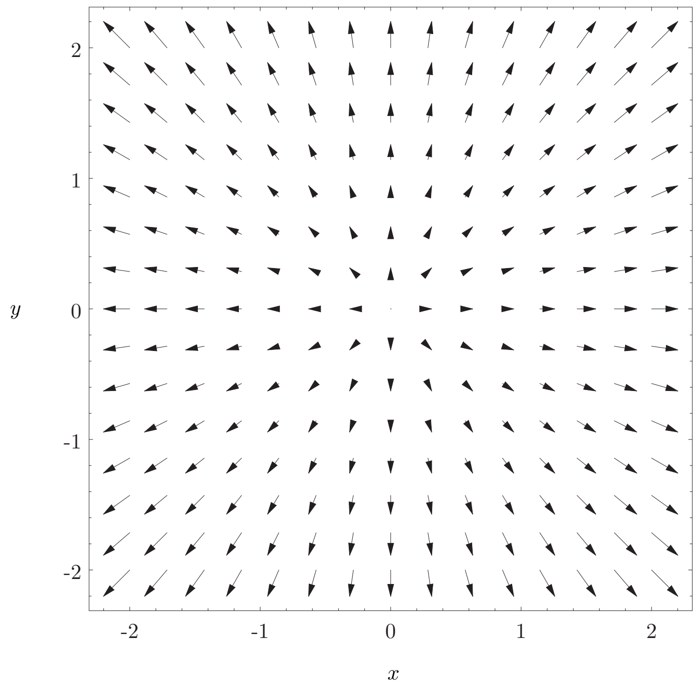

Figure 3.3: Example of a vector field with positive divergence (everywhere): $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$.

3.4 The Curl of a vector field

(See Thomas 16.7)
The curl of a vector field $\mathbf{F}$ is defined to be

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}. \tag{3.2}$$

Note that curl $\mathbf{F}$ is a vector, since there are $\mathbf{i}, \mathbf{j}, \mathbf{k}$ on the RHS; and it generally depends on position so it's a new **vector** field.

We can write $\nabla \times \mathbf{F}$ as curl $\mathbf{F}$ – again the two notations are completely interchangeable. It is convenient to remember $\nabla \times \mathbf{F}$ in terms of a determinant like the one for $\mathbf{v} \times \mathbf{w}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

It is easy to verify, by writing out the determinant in full, that this is equivalent to the original definition.

It is also easy to show, by writing out the components, that if $\mathbf{F}, \mathbf{G}$ are any two vector fields,

$$\nabla \times (\mathbf{F} + \mathbf{G}) = (\nabla \times \mathbf{F}) + (\nabla \times \mathbf{G})$$

and if λ is any constant then

$$\nabla \times (\lambda \mathbf{F}) = \lambda (\nabla \times \mathbf{F})$$

Note that the equality above **only** works if λ is a constant (independent of x, y, z): see the next section for more general products.

The geometrical meaning of the curl is as follows. Loosely speaking, if at some point in space the component of the curl in the $\mathbf{n}$ direction is positive, it means that in the vicinity of the point and in a plane normal to $\mathbf{n}$, the vector field tends to go round in an anticlockwise direction if one looks along vector $\mathbf{n}$. If the component of the curl were negative, it would mean that the vector field tends to go round in a clockwise direction. (See Fig. 3.4.) This idea will be made more precise when we come to Stokes's Theorem.

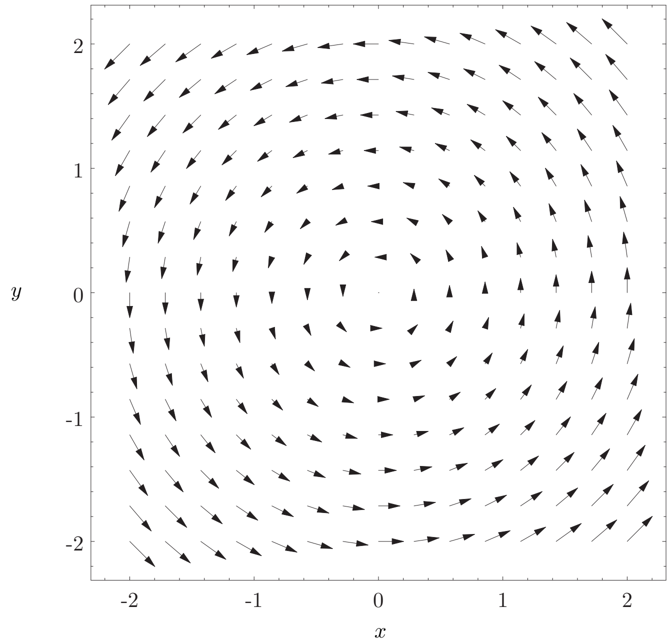

Figure 3.4: Example of a vector field with positive curl (in the z direction): $\mathbf{F} = x\mathbf{j} - y\mathbf{i}$.

A vector field $\mathbf{F}$ for which $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere is called *curl-free* or *irrotational*.

Example 3.4. The velocity in a fluid is $\mathbf{v} = y\mathbf{i} - x\mathbf{j} + 0\mathbf{k}$. Find $\nabla \times \mathbf{v}$.

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{vmatrix} = \mathbf{i}(0-0) + \mathbf{j}(0-0) + \mathbf{k}(-1-1) = -2\mathbf{k}.$$

Exercise 3.3. If $\mathbf{F} = (x^2+y^2+z^2)\mathbf{i} + (x^4 - y^2z^2)\mathbf{j} + xyz\mathbf{k}$, find $\nabla \times \mathbf{F}$. □

Exercise 3.4. Find the divergence $(\nabla \cdot \mathbf{F})$ and curl $(\nabla \times \mathbf{F})$ of the following vector fields:

$\mathbf{F} = x^2\mathbf{i} + xz\mathbf{j} - 3z\mathbf{k}$

$\mathbf{F} = x^2\mathbf{i} - 2xy\mathbf{j} + 3xz\mathbf{k}$

$\mathbf{F} = \nabla(1/r)$ where $r = (x^2+y^2+z^2)^{1/2} \neq 0$.

□

3.5 Grad, Div and Curl of products

(See Thomas 16.7 and the exercises to 16.8)
We can now consider the application of grad, div and curl to products. We saw above that grad, div and curl behave in the "obvious" way for addition and multiplication by a constant.

However, we can also multiply scalar and/or vector fields together (in pairs) to get new scalar and vector fields. Altogether there are four ways to do this, as follows: if we have two scalar fields $U(\mathbf{r}), V(\mathbf{r})$ we can ordinary-multiply them (at each point $\mathbf{r}$) to get a new scalar field $UV = U(\mathbf{r})V(\mathbf{r})$; likewise for a scalar field $U(\mathbf{r})$ and a vector field $\mathbf{F}(\mathbf{r})$ we can use ordinary multiplication to give $U\mathbf{F} \equiv U(\mathbf{r})\mathbf{F}(\mathbf{r})$; the value is a vector, so this is a vector field. Also, if we have two vector fields $\mathbf{F}, \mathbf{G}$ we can define their dot product $\mathbf{F}.\mathbf{G} \equiv (\mathbf{F}(\mathbf{r}) \cdot (\mathbf{G}(\mathbf{r}))$ and their cross product $\mathbf{F} \times \mathbf{G}$ in the obvious way, by taking the dot or cross products of each field at the **same** point $\mathbf{r}$. Clearly $\mathbf{F} \cdot \mathbf{G}$ is a scalar field, and $\mathbf{F} \times \mathbf{G}$ is a vector field.

Note: in each of these products, the values of $U, V, \mathbf{F}, \mathbf{G}$ are taken at the **same** point $\mathbf{r}$ in the product. In longer equations, it is common to not bother writing in all the $(\mathbf{r})$'s, because if something is defined as a field then we know it is a function of $\mathbf{r}$.

We can now apply grad, div and curl to these products, but only for the following allowed combinations: to apply grad, we have to have a product which is itself a scalar field: that can be either an ordinary multiple of two scalar fields, say UV, or a scalar product (dot product) of two vector fields, $\mathbf{F} \cdot \mathbf{G}$.

Div and curl can only be applied to a vector field, so the possible products we could have look like $U\mathbf{F}$ or the cross product $\mathbf{F} \times \mathbf{G}$ above.

If we were dealing with functions of a single variable, the derivative would just give the well-known product rule for derivatives,

$$\frac{\mathrm{d}(fg)}{\mathrm{d}x} = f\frac{\mathrm{d}g}{\mathrm{d}x} + \frac{\mathrm{d}f}{\mathrm{d}x}g. \tag{3.3}$$

Some of the vector cases are just like that, but some are more complicated: we next give the results, and discuss the details afterwards. There are six cases as we've outlined above (two each for grad, div and curl).

For grad of products we have:

$$\begin{aligned} \nabla(UV) &= U(\nabla V) + V(\nabla U) && (3.4)\\ \text{or} \qquad \operatorname{grad}(UV) &= U \operatorname{grad} V + V \operatorname{grad} U \\ \nabla(\mathbf{F}\cdot\mathbf{G}) &= \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}) + (\mathbf{F}.\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} && (3.5) \end{aligned}$$

For div of products we have:

$$\begin{aligned} \nabla\cdot(U\mathbf{F}) &= U(\nabla\cdot\mathbf{F}) + (\nabla U).\mathbf{F} && (3.6)\\ \nabla\cdot(\mathbf{F}\times\mathbf{G}) &= \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G}) && (3.7) \end{aligned}$$

and for curl of products, we have:

$$\begin{aligned} \nabla\times(U\mathbf{F}) &= U(\nabla\times\mathbf{F}) + (\nabla U)\times\mathbf{F} && (3.8)\\ &= U(\nabla\times\mathbf{F}) - \mathbf{F}\times(\nabla U) \\ \nabla\times(\mathbf{F}\times\mathbf{G}) &= \mathbf{F}(\nabla\cdot\mathbf{G}) + (\mathbf{G}.\nabla)\mathbf{F} - \mathbf{G}(\nabla\cdot\mathbf{F}) - (\mathbf{F}.\nabla)\mathbf{G} && (3.9) \end{aligned}$$

We see above that equations 3.4, 3.6, 3.7 and 3.8 look quite similar to 3.3, except for the minus sign in 3.7 and the possible minus sign in 3.8.

Note also that Eqs. 3.4 and 3.5 are symmetrical in the two variables, while 3.7 and 3.9 are antisymmetric, i.e. they must change sign if $\mathbf{F},\mathbf{G}$ are swapped, due to the antisymmetry of the cross product.

Note: if you set $U(x,y,z) = \lambda$ =constant in the above, that is a (very boring) but legal scalar field with $\nabla U = \mathbf{0}$ everywhere; then you'll see Eqs. 3.4, 3.6, 3.8 reduce to the obvious cases of multiplication by a constant which we've met before. But for multiplication by a non-constant scalar U, the second terms involving ∇U appear on the RHS.

The other two equations 3.5, 3.9 are more complicated, and involve the new operator $(\mathbf{G}.\nabla)$: this is defined so for a scalar field V, if $\mathbf{G} = (G_1, G_2, G_3)$,

$$(\mathbf{G}.\nabla)V = \left(G_1\frac{\partial}{\partial x} + G_2\frac{\partial}{\partial y} + G_3\frac{\partial}{\partial z}\right)V = G_1\frac{\partial V}{\partial x} + G_2\frac{\partial V}{\partial y} + G_3\frac{\partial V}{\partial z} \ ,$$

For a vector field $\mathbf{F}$, the notation $(\mathbf{G}.\nabla)\mathbf{F}$ is to be interpreted as $(\mathbf{G}.\nabla F_1, \mathbf{G}.\nabla F_2, \mathbf{G}.\nabla F_3)$, taking $\mathbf{F} = (F_1, F_2, F_3)$; Thus writing out the whole thing, we have

$$(\mathbf{G}.\nabla)\mathbf{F} = \left(G_1\frac{\partial F_1}{\partial x} + G_2\frac{\partial F_1}{\partial y} + G_3\frac{\partial F_1}{\partial z},\ G_1\frac{\partial F_2}{\partial x} + G_2\frac{\partial F_2}{\partial y} + G_3\frac{\partial F_2}{\partial z},\ G_1\frac{\partial F_3}{\partial x} + G_2\frac{\partial F_3}{\partial y} + G_3\frac{\partial F_3}{\partial z}\right)$$

This is essentially the directional derivative of vector $\mathbf{F}$ in the direction of $\mathbf{G}$, i.e. it is $|\mathbf{G}|$ times the derivative $d\mathbf{F}/ds$ along the direction of the unit vector parallel to $\mathbf{G}$.

(Warning: the form of this definition will not persist in curvilinear coordinates, but the directional derivative will remain the same).

Note: you are not expected to memorise Eqs. 3.5 and 3.9, but you may be given those formulae in an exam question. You should know the definition of $(\mathbf{G}.\nabla)\mathbf{F}$ above.

Example 3.5. Let $\mathbf{a}$ be a constant vector, and $r = |\mathbf{r}|$ as usual. Then, using Eq 3.8,

$$\begin{aligned} \nabla \times (r\mathbf{a}) &= r(\nabla \times \mathbf{a}) - \mathbf{a} \times \nabla r \\ &= \mathbf{0} - \frac{\mathbf{a} \times \mathbf{r}}{r} \end{aligned}$$

since the curl of a constant $\mathbf{a}$ is zero, and $\nabla r = \mathbf{r}/r$ (as in Coursework 2).

Example 3.6. Let $\mathbf{a}$ be a constant vector. Then, using Equation 3.9,

$$\begin{aligned} \nabla \times (\mathbf{a} \times \mathbf{r}) &= \mathbf{a}(\nabla \cdot \mathbf{r}) + (\mathbf{r}.\nabla)\mathbf{a} - \mathbf{r}(\nabla \cdot \mathbf{a}) - (\mathbf{a}.\nabla)\mathbf{r} \\ &= 3\mathbf{a} + \mathbf{0} - \mathbf{0} - \mathbf{a} \\ &= 2\mathbf{a} \end{aligned}$$

(On the top line, the two middle terms differentiate the constant $\mathbf{a}$ so are both zero, and it is simple to check from the definitions that $\nabla \cdot \mathbf{r} = 3$ and $(\mathbf{a}.\nabla)\mathbf{r} = \mathbf{a}$.)

Proofs:

All of the equations 3.4 to 3.9 can be proved directly from the definitions by inserting components, expanding out using the ordinary derivative-of-product rule and doing some rearrangement; this can be fairly long, but is not difficult.

For a couple of examples: firstly for Eq. 3.4 it is simple, we have

$$\nabla(UV) \quad = \quad \mathbf{i}\frac{\partial}{\partial x}(UV) + \mathbf{j}\frac{\partial}{\partial y}(UV) + \mathbf{k}\frac{\partial}{\partial z}(UV)$$

$$
\begin{aligned}
&= \mathbf{i}(U\frac{\partial V}{\partial x} + V\frac{\partial U}{\partial x}) + \mathbf{j}(U\frac{\partial V}{\partial y} + V\frac{\partial U}{\partial y}) + \mathbf{k}(U\frac{\partial V}{\partial z} + V\frac{\partial U}{\partial z}) \\
&= U\left(\mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial y} + \mathbf{k}\frac{\partial V}{\partial z}\right) + V\left(\mathbf{i}\frac{\partial U}{\partial x} + \mathbf{j}\frac{\partial U}{\partial y} + \mathbf{k}\frac{\partial U}{\partial z}\right) \\
&= U(\nabla V) + V(\nabla U) \qquad \text{QED.}
\end{aligned}
$$

Next we'll prove Eq.3.8: the product $U\mathbf{F}$ is a vector field with components (UF_1, UF_2, UF_3); inserting those into the definition of curl,

$$
\begin{aligned}
\nabla \times (U\mathbf{F}) &= \mathbf{i}\left(\frac{\partial}{\partial y}(UF_3) - \frac{\partial}{\partial z}(UF_2)\right) + \mathbf{j}\left(\frac{\partial}{\partial z}(UF_1) - \frac{\partial}{\partial x}(UF_3)\right) + \mathbf{k}\left(\frac{\partial}{\partial x}(UF_2) - \frac{\partial}{\partial y}(UF_1)\right) \\
&= \mathbf{i}\left(U\frac{\partial F_3}{\partial y} + F_3\frac{\partial U}{\partial y} - U\frac{\partial F_2}{\partial z} - F_2\frac{\partial U}{\partial z}\right) + \mathbf{j}\left(U\frac{\partial F_1}{\partial z} + F_1\frac{\partial U}{\partial z} - U\frac{\partial F_3}{\partial x} - F_3\frac{\partial U}{\partial x}\right) \\
&\quad + \mathbf{k}\left(U\frac{\partial F_2}{\partial x} + F_2\frac{\partial U}{\partial x} - U\frac{\partial F_1}{\partial y} - F_1\frac{\partial U}{\partial y}\right)
\end{aligned}
$$

Now we just re-order the 12 terms so that the six with a $U\partial F_i$ come first, then the six with an $F_i\partial U$ come next; and from the definitions, it becomes clear that the result is

$$
\nabla \times (U\mathbf{F}) = U(\nabla \times \mathbf{F}) + (\nabla U) \times \mathbf{F} \qquad \text{QED.}
$$

The others can be proved in a similar way, though it gets quite long for Eqs. 3.5 and 3.9. Much shorter proofs can be given using **index notation**, but this is no longer on the syllabus.

Note: As always, be careful what is a scalar and what is a vector. Remember that in an equation

$$
(\textit{expression } 1) = (\textit{expression } 2) + (\textit{expression } 3) + \ldots
$$

expressions 1,2,3 ... must be either all scalars or all vectors, since you cannot add a scalar and a vector. Check that you understand that in the above equations, all the expressions are vectors for 3.4, 3.5, 3.8, 3.9 (because grad() or curl() give a vector result); while they are scalars for 3.6 and 3.7 because div gives a scalar result.

3.6 Vector second derivatives: applying ∇ twice

We also have a second set of identities arising from applying **two** of grad, div or curl in succession. Here grad U and curl **F** produce vector fields, to which either div or curl can be applied; while div **F** produces a scalar field, and then we can apply grad to that. This gives a total of five allowed cases, which are as follows:

$$
\begin{aligned}
\text{div(grad U)} &= \nabla \cdot (\nabla U) \equiv \nabla^2 U && (3.10) \\
\text{curl}(\text{grad}\, U) &= \nabla \times (\nabla U) = \mathbf{0} && (3.11) \\
\text{div}(\text{curl}\, \mathbf{F}) &= \nabla \cdot (\nabla \times \mathbf{F}) = 0 && (3.12) \\
\text{curl}(\text{curl}\, \mathbf{F}) &= \nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} && (3.13) \\
\text{grad}(\text{div}\, \mathbf{F}) &= \nabla(\nabla \cdot \mathbf{F}) = \nabla \times (\nabla \times \mathbf{F}) + \nabla^2 \mathbf{F} && (3.14)
\end{aligned}
$$

We see here that two of these cases (curl grad U, and div curl **F**) are identically zero ; this is true for any fields, as long as they are sufficiently well behaved that the partial derivatives commute, see below. These two zero cases can be helpfully memorised by the fact that they would also give zero if ∇ was replaced by an ordinary vector $\mathbf{a}$; but beware, this sort of rule is not applicable to every equation containing ∇.

The first equation above Eq. 3.10 introduces a new operator ∇^2 called the **Laplacian**; this is very important in a wide range of physical problems, and we will meet it extensively in Chapter 7. In components, combining the definition of grad U from Chapter 1 and plugging that into Eq. 3.1, we get simply

$$\nabla^2 U \equiv \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \tag{3.15}$$

This Laplacian operator can be applied to either a scalar field or a vector field, producing a field of the same type; in the above, U is a scalar field and $\nabla^2 U$ is another scalar field.

For a vector field $\mathbf{F}$, to get $\nabla^2\mathbf{F}$ we apply ∇^2 to each component of $\mathbf{F}$ separately, giving

$$\nabla^2\mathbf{F} = \mathbf{i}\nabla^2 F_1 + \mathbf{j}\nabla^2 F_2 + \mathbf{k}\nabla^2 F_3$$

so $\nabla^2\mathbf{F}$ is another vector field.

Also note that the last two of the above equations 3.13 and 3.14 are just a rearrangement of each other, giving a relationship between curl curl $\mathbf{F}$, grad div $\mathbf{F}$ and $\nabla^2\mathbf{F}$.

All of the relations above can be proved by direct substitution, e.g.:

Proof of 3.11:

$$\begin{aligned}
\mathrm{curl}(\nabla U) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial U/\partial x & \partial U/\partial y & \partial U/\partial z \end{vmatrix} \\
&= \left(\frac{\partial^2 U}{\partial y \partial z} - \frac{\partial^2 U}{\partial z \partial y},\ \frac{\partial^2 U}{\partial z \partial x} - \frac{\partial^2 U}{\partial x \partial z},\ \frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 U}{\partial y \partial x} \right) = 0.
\end{aligned}$$

[Note, we assume that the function U is sufficiently well-behaved for its partial second derivatives to commute.]

The relation curl grad $U = \mathbf{0}$ is particularly useful, since it is often interesting to ask, given some vector field $\mathbf{F}$, can we find a scalar field U such that $\nabla U = \mathbf{F}$? If we can, this simplifies things from 3 functions of position to 1 function.

Now we can show that if our given $\mathbf{F}$ has curl $\mathbf{F} \neq \mathbf{0}$, it is not possible to find such a scalar field U, as follows: choose any scalar field U, and define a vector field $\mathbf{H} = \nabla U$. We'd like to find a U such that $\mathbf{H} = \mathbf{F}$. But from Eq. 3.11, $\nabla \times \mathbf{H} = \nabla \times (\nabla U) = \mathbf{0}$. Therefore $\mathbf{H} \neq \mathbf{F}$: so, if curl $\mathbf{F} \neq \mathbf{0}$ then it is **not** possible to express $\mathbf{F}$ as the gradient of any scalar field U.

The converse is also true: we will show in the next chapter that if curl $\mathbf{F} = \mathbf{0}$ everywhere in a given domain, then we *can* find a scalar field U with $\nabla U = \mathbf{F}$: and we'll also show how to construct the desired U using a suitable integral. This requires vector integration, which we'll do in the next Chapter.