3.5 GRAD, DIV & CURL OF PRODUCTS.

First we need to define the various sorts of product.

Suppose we are given
two scalar fields U, V
and two vector fields $\underset{\sim}{F}, \underset{\sim}{G}$.

and the operations
ordinary multiplication $\left\{ \begin{array}{l} \text{two scalar fields} \\ \text{one scalar \& one vector f.} \end{array} \right.$
dot-product (of two vectors)
cross-product (of two vectors)

We can get four types of product:

1) Multiply U and V:
get scalar field $UV \equiv U(\underset{\sim}{r})V(\underset{\sim}{r})$

2) Multiply U and $\underset{\sim}{F}$
get vector field $U\underset{\sim}{F} \equiv U(\underset{\sim}{r})\underset{\sim}{F}(\underset{\sim}{r})$

3) Dot-product of $\underset{\sim}{F}, \underset{\sim}{G}$:
get scalar field $\underset{\sim}{F}\cdot\underset{\sim}{G} \equiv \underset{\sim}{F}(\underset{\sim}{r})\cdot\underset{\sim}{G}(\underset{\sim}{r})$

4) Cross-product of $\underset{\sim}{F}$ and $\underset{\sim}{G}$:

get vector field

$$\underset{\sim}{F} \times \underset{\sim}{G} \equiv \left(\underset{\sim}{F}(\underset{\sim}{r})\right) \times \left(\underset{\sim}{G}(\underset{\sim}{r})\right).$$

[Note: $V\underset{\sim}{F}$, $U\underset{\sim}{G}$ etc are same 'type' as $U\underset{\sim}{F}$, so don't count separately.]

Of the above :

UV, $\underset{\sim}{F} \cdot \underset{\sim}{G}$ are scalar fields

- can take grad.

$U\underset{\sim}{F}$, $\underset{\sim}{F} \times \underset{\sim}{G}$ are vector fields

- can take div or curl.

This gives us six cases.

Also, remember the simple case

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g.$$

For grad of products we have:

$$
\begin{aligned}
\nabla(UV) &= U(\nabla V) + V(\nabla U) && (3.4)\\
\text{or} \quad \operatorname{grad}(UV) &= U \operatorname{grad} V + V \operatorname{grad} U\\
\nabla(\mathbf{F}\cdot\mathbf{G}) &= \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}) + (\mathbf{F}.\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} && (3.5)
\end{aligned}
$$

For div of products we have:

$$
\begin{aligned}
\nabla\cdot(U\mathbf{F}) &= U(\nabla\cdot\mathbf{F}) + (\nabla U).\mathbf{F} && (3.6)\\
\nabla\cdot(\mathbf{F}\times\mathbf{G}) &= \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G}) && (3.7)
\end{aligned}
$$

and for curl of products, we have:

$$
\begin{aligned}
\nabla\times(U\mathbf{F}) &= U(\nabla\times\mathbf{F}) + (\nabla U)\times\mathbf{F} && (3.8)\\
&= U(\nabla\times\mathbf{F}) - \mathbf{F}\times(\nabla U)\\
\nabla\times(\mathbf{F}\times\mathbf{G}) &= \mathbf{F}(\nabla\cdot\mathbf{G}) + (\mathbf{G}.\nabla)\mathbf{F} - \mathbf{G}(\nabla\cdot\mathbf{F}) - (\mathbf{F}.\nabla)\mathbf{G} && (3.9)
\end{aligned}
$$

We see above that equations 3.4, 3.6, 3.7 and 3.8 look quite similar to 3.3, except for the minus sign in 3.7 and the possible minus sign in 3.8.

Take care what's a scalar and what's a vector.

You <u>cannot</u> <u>add</u> scalar + vector,

so any equation like

(Expression 1) = (Expression 2) + (Expression 3) + ...

Expression 1, 2, 3 ...
must be the same type
(all scalars or all vectors).

Remember:

grad : U is a scalar, $\underset{\sim}{\nabla} U$ is a vector

div : $\underset{\sim}{F}$ is a vector, $\underset{\sim}{\nabla} \cdot \underset{\sim}{F}$ is a scalar

curl : $\underset{\sim}{F}$ is a vector, $\underset{\sim}{\nabla} \times \underset{\sim}{F}$ is a vector.

$$\text{div}(U\underset{\sim}{F}) = U(\text{div}\,\underset{\sim}{F}) + (\text{grad}\,U)\cdot\underset{\sim}{F}$$

Eq. 3.6 :

$$\underbrace{\underset{\sim}{\nabla}\cdot(U\underbrace{\underset{\sim}{F}}_{\text{vector}})}_{\substack{\text{div (vector)} \\ = \text{scalar}}} = \underset{\text{scalar}}{U}\,(\underbrace{\underset{\sim}{\nabla}\cdot\underset{\sim}{F}}_{\text{scalar}}) + \underbrace{(\underset{\text{vector}}{\underset{\sim}{\nabla}U})\cdot\underset{\text{vector}}{\underset{\sim}{F}}}_{\substack{\text{dot product} \\ \text{is scalar.}}}$$

scalar Field = scalar field + scalar Field.

Eq. 3.8

$$\underbrace{\underset{\sim}{\nabla}\times(U\underbrace{\underset{\sim}{F}}_{\text{vector}})}_{\substack{\text{curl (vector)} \\ = \text{vector}}} = \underbrace{\underset{\text{scalar}}{U}\,(\underbrace{\underset{\sim}{\nabla}\times\underset{\sim}{F}}_{\text{vector}})}_{\text{vector}} + \underbrace{(\underbrace{\underset{\sim}{\nabla}U}_{\text{vector}})\times\underset{\text{vector}}{\underset{\sim}{F}}}_{\substack{\text{cross product} \\ \text{is vector.}}}$$

Proof 3.4 :

For a couple of examples: firstly for Eq. 3.4 it is simple, we have

$$
\begin{aligned}
\nabla(UV) &= \mathbf{i}\frac{\partial}{\partial x}(UV)+\mathbf{j}\frac{\partial}{\partial y}(UV)+\mathbf{k}\frac{\partial}{\partial z}(UV) \\
&= \mathbf{i}(U\frac{\partial V}{\partial x}+V\frac{\partial U}{\partial x})+\mathbf{j}(U\frac{\partial V}{\partial y}+V\frac{\partial U}{\partial y})+\mathbf{k}(U\frac{\partial V}{\partial z}+V\frac{\partial U}{\partial z}) \\
&= U\left(\mathbf{i}\frac{\partial V}{\partial x}+\mathbf{j}\frac{\partial V}{\partial y}+\mathbf{k}\frac{\partial V}{\partial z}\right)+V\left(\mathbf{i}\frac{\partial U}{\partial x}+\mathbf{j}\frac{\partial U}{\partial y}+\mathbf{k}\frac{\partial U}{\partial z}\right) \\
&= U(\nabla V)+V(\nabla U) \qquad \text{QED.}
\end{aligned}
$$

Product rule

Rearrange.

Next we'll prove Eq.3.8: the product $U\mathbf{F}$ is a vector field with components (UF_1, UF_2, UF_3); inserting those into the definition of curl,

$$
\begin{aligned}
\nabla\times(U\mathbf{F}) &= \mathbf{i}\left(\frac{\partial}{\partial y}(UF_3)-\frac{\partial}{\partial z}(UF_2)\right)+\mathbf{j}\left(\frac{\partial}{\partial z}(UF_1)-\frac{\partial}{\partial x}(UF_3)\right)+\mathbf{k}\left(\frac{\partial}{\partial x}(UF_2)-\frac{\partial}{\partial y}(UF_1)\right) \\
&= \mathbf{i}\left(U\frac{\partial F_3}{\partial y}+F_3\frac{\partial U}{\partial y}-U\frac{\partial F_2}{\partial z}-F_2\frac{\partial U}{\partial z}\right)+\mathbf{j}\left(U\frac{\partial F_1}{\partial z}+F_1\frac{\partial U}{\partial z}-U\frac{\partial F_3}{\partial x}-F_3\frac{\partial U}{\partial x}\right)
\end{aligned}
$$

Next we'll prove Eq.3.8: the product $U\mathbf{F}$ is a vector field with components (UF_1, UF_2, UF_3); inserting those into the definition of curl,

$$\begin{aligned}
\nabla \times (U\mathbf{F}) &= \mathbf{i}\left(\frac{\partial}{\partial y}(UF_3) - \frac{\partial}{\partial z}(UF_2)\right) + \mathbf{j}\left(\frac{\partial}{\partial z}(UF_1) - \frac{\partial}{\partial x}(UF_3)\right) + \mathbf{k}\left(\frac{\partial}{\partial x}(UF_2) - \frac{\partial}{\partial y}(UF_1)\right) \\
&= \mathbf{i}\left(U\frac{\partial F_3}{\partial y} + F_3\frac{\partial U}{\partial y} - U\frac{\partial F_2}{\partial z} - F_2\frac{\partial U}{\partial z}\right) + \mathbf{j}\left(U\frac{\partial F_1}{\partial z} + F_1\frac{\partial U}{\partial z} - U\frac{\partial F_3}{\partial x} - F_3\frac{\partial U}{\partial x}\right) \\
&\quad + \mathbf{k}\left(U\frac{\partial F_2}{\partial x} + F_2\frac{\partial U}{\partial x} - U\frac{\partial F_1}{\partial y} - F_1\frac{\partial U}{\partial y}\right)
\end{aligned}$$

Product rule

Re-order

$$= U\left[\underset{\sim}{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \underset{\sim}{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \underset{\sim}{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$+ \left[\underset{\sim}{i}\left(F_3\frac{\partial U}{\partial y} - F_2\frac{\partial U}{\partial z}\right) + \underset{\sim}{j}\left(F_1\frac{\partial U}{\partial z} - F_3\frac{\partial U}{\partial x}\right) + \underset{\sim}{k}\left(F_2\frac{\partial U}{\partial x} - F_1\frac{\partial U}{\partial y}\right)\right]$$

$$= U\left[\operatorname{curl} \underset{\sim}{F}\right] + \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\underset{\sim}{\nabla} \times (U\underset{\sim}{F}) = \ ?$$

$$U\underset{\sim}{F} \equiv UF_1\,\underset{\sim}{i} + UF_2\,\underset{\sim}{j} + UF_3\,\underset{\sim}{k}.$$

$$\underset{\sim}{\nabla} \times \cancel{F}(U\underset{\sim}{F}) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ UF_1 & UF_2 & UF_3 \end{vmatrix}$$

$(\underset{\sim}{G} . \underset{\sim}{\nabla})$ is a new operator :

from $\underset{\sim}{G} = (G_1, G_2, G_3)$

and $\underset{\sim}{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

get $(\underset{\sim}{G} . \underset{\sim}{\nabla}) = G_1 \frac{\partial}{\partial x} + G_2 \frac{\partial}{\partial y} + G_3 \frac{\partial}{\partial z}$

so $$(\underset{\sim}{G} . \underset{\sim}{\nabla}) \underset{\sim}{F} = \underset{\sim}{i} \left(G_1 \frac{\partial F_1}{\partial x} + G_2 \frac{\partial F_1}{\partial y} + G_3 \frac{\partial F_1}{\partial z} \right) + \underset{\sim}{j} \left(G_1 \frac{\partial F_2}{\partial x} + \dots \right) + \underset{\sim}{k} (\dots) .$$

This is geometrically $|G|$ times the directional derivative of $\underset{\sim}{F}$ along a unit vector parallel to $\underset{\sim}{G}$.

Example 3.5:

Let $\underset{\sim}{a}$ be a constant vector, and $r = |\underset{\sim}{r}|$ as usual.

What is $\operatorname{curl}(r\underset{\sim}{a})$?

r is scalar; $\underset{\sim}{a}$ is vector

so need Eq. 3.8:

$$\underset{\sim}{\nabla} \times (r\underset{\sim}{a}) = r(\underset{\sim}{\nabla} \times \underset{\sim}{a}) - \underset{\sim}{a} \times (\underset{\sim}{\nabla} r)$$

$$= r\,\underset{\sim}{0} - \underset{\sim}{a} \times \frac{1}{r}\underset{\sim}{r}$$

using Coursework 2.

Example 3.6:

Let $\underset{\sim}{a}$ be a constant vector; ~~and~~

evaluate $\underset{\sim}{\nabla} \times (\underset{\sim}{a} \times \underset{\sim}{r})$.

Eq. 3.9:

$$\underset{\sim}{\nabla} \times (\underset{\sim}{a} \times \underset{\sim}{r}) = \underset{\sim}{a}(\underset{\sim}{\nabla} \cdot \underset{\sim}{r}) + (\underset{\sim}{r} \cdot \underset{\sim}{\nabla})\underset{\sim}{a}$$

$$\cancel{+} - \underset{\sim}{r}(\underset{\sim}{\nabla} \cdot \underset{\sim}{a}) - (\underset{\sim}{a} \cdot \underset{\sim}{\nabla})\underset{\sim}{r}$$

Now $\underset{\sim}{\nabla} \cdot \underset{\sim}{r} = 3$

$\underset{\sim}{a}$ const $\Rightarrow (\underset{\sim}{r} \cdot \underset{\sim}{\nabla})\underset{\sim}{a} = 0$, $(\underset{\sim}{\nabla} \cdot \underset{\sim}{a}) = 0$.

$$\underset{\sim}{r} = (x, y, z)$$

$$\text{so} \quad (\underset{\sim}{a} \cdot \underset{\sim}{\nabla})\underset{\sim}{r} = \underset{\sim}{i}\left(a_1 \frac{\partial x}{\partial x} + a_2 \cancel{\frac{\partial x}{\partial y}} + a_3 \cancel{\frac{\partial x}{\partial z}}\right)$$

$$+ \underset{\sim}{j}\left(a_1 \cancel{\frac{\partial x}{\partial y}} + \quad \cdots \quad \right)$$

$$= \underset{\sim}{i}\, a_1 + \underset{\sim}{j}\, a_2 + \underset{\sim}{k}\, a_3$$

$$= \underset{\sim}{a}\,.$$

$$\text{So} \quad \underset{\sim}{\nabla} \times (\underset{\sim}{a} \times \underset{\sim}{r}) = 3\underset{\sim}{a} - \underset{\sim}{a}$$

$$= 2\underset{\sim}{a}\,.$$

CWK 2: $\nabla(r^n) = n r^{n-2} \underset{\sim}{r}$

$= n r^{n-1} \hat{\underset{\sim}{r}}$.

$n = 1$:

$\nabla(r) = 1 r^{-1} \underset{\sim}{r}$.

$= \frac{1}{r} \underset{\sim}{r} = \hat{\underset{\sim}{r}}$.

3.6 VECTOR SECOND DERIVATIVES.

There are another set of formulae arising from applying TWO of grad, div, or curl in succession.

If we start with scalar field U and vector field $\underset{\sim}{F}$, the allowed first derivatives are

$$\text{grad } U \equiv \underset{\sim}{\nabla} U \quad : \text{ vector field}$$

$$\text{div } \underset{\sim}{F} \equiv \underset{\sim}{\nabla} \cdot \underset{\sim}{F} \quad : \text{ scalar field}$$

$$\text{curl } \underset{\sim}{F} \equiv \underset{\sim}{\nabla} \times \underset{\sim}{F} \quad : \text{ vector field.}$$

Now grad U, curl $\underset{\sim}{F}$ are vector fields
$\Rightarrow$ can apply div <u>or</u> curl.

div $\underset{\sim}{F}$ is a scalar field $\Rightarrow$ can apply grad.

This gives us $2 + 2 + 1 = 5$ cases, which are 3.10 – 3.14.

grad() or curl() give a vector result); while they are scalars for 3.6 and 3.7 because div gives a scal

3.6 Vector second derivatives: applying ∇ twice

We also have a second set of identities arising from applying **two** of grad, div or curl in succession. U and curl **F** produce vector fields, to which either div or curl can be applied; while div **F** produce field, and then we can apply grad to that. This gives a total of five allowed cases, which are as follo

$$\begin{aligned}
\text{div}(\text{grad U}) &= \nabla\cdot(\nabla U) \equiv \nabla^2 U && (3.10)\\
\text{curl}(\text{grad } U) &= \nabla\times(\nabla U) = \mathbf{0} && (3.11)\\
\text{div}(\text{curl } \mathbf{F}) &= \nabla\cdot(\nabla\times\mathbf{F}) = 0 && (3.12)\\
\text{curl}(\text{curl } \mathbf{F}) &= \nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F} && (3.13)\\
\text{grad}(\text{div } \mathbf{F}) &= \nabla(\nabla\cdot\mathbf{F}) = \nabla\times(\nabla\times\mathbf{F}) + \nabla^2\mathbf{F} && (3.14)
\end{aligned}$$

We see here that two of these cases (curl grad U, and div curl **F**) are identically zero ; this is tru fields, as long as they are sufficiently well behaved that the partial derivatives commute, see belo

3.10 : $\nabla \cdot (\underset{\sim}{\nabla U}) \equiv \mathrm{div}\,(\mathrm{grad}\ U)$

$\equiv \nabla^2 U$

↑

LAPLACIAN .

In components,

$$\mathrm{grad}\ U = \frac{\partial U}{\partial x}\,\underset{\sim}{i} + \frac{\partial U}{\partial y}\,\underset{\sim}{j} + \frac{\partial U}{\partial z}\,\underset{\sim}{k}$$

$$\nabla^2 U = \mathrm{div}\,(\mathrm{grad}\ U) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} .$$

$$\Rightarrow \quad \nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

Above, U is a scalar field,

~~&~~ grad U is a vector field,

$\nabla^2 U \equiv \mathrm{div}\,(\mathrm{grad}\ U)$ is a scalar field.

In (3.13), (3.14), we can also apply ∇^2 to a <u>vector</u> field and get another <u>vector</u> field :

if $\underset{\sim}{F} = F_1\,\underset{\sim}{i} + F_2\,\underset{\sim}{j} + F_3\,\underset{\sim}{k}$,

$$\nabla^2 \underset{\sim}{F} \equiv \nabla^2 F_1\,\underset{\sim}{i} + \nabla^2 F_2\,\underset{\sim}{j} + \nabla^2 F_3\,\underset{\sim}{k} .$$

$$\underset{\sim}{i}\left(\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial y}\right)\right)$$

$$+\ \underset{\sim}{j}\left(\ \cdots\ \right)$$

$$\underset{\sim}{i}\left(\frac{\partial^2 U}{\partial y\,\partial z}-\frac{\partial^2 U}{\partial z\,\partial y}\right)+\cdots$$

$= \underset{\sim}{0}$ as long as partial second derivatives 'commute'.

$$\frac{\partial^2 U}{\partial y\,\partial z}=\frac{\partial^2 U}{\partial z\,\partial y}.$$

3.11 curl (grad U) $= \underset{\sim}{0}$

$$\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} U) = \underset{\sim}{0}$$

$$\underset{\sim}{\nabla} U = \frac{\partial U}{\partial x}\underset{\sim}{i} + \frac{\partial U}{\partial y}\underset{\sim}{j} + \frac{\partial U}{\partial z}\underset{\sim}{k}.$$

All of the relations above can be proved by direct substitution, e.g.:

Proof of 3.11:

$$\begin{aligned}
\text{curl}(\nabla U) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial U/\partial x & \partial U/\partial y & \partial U/\partial z \end{vmatrix} \\
&= \left(\frac{\partial^2 U}{\partial y \partial z} - \frac{\partial^2 U}{\partial z \partial y}, \frac{\partial^2 U}{\partial z \partial x} - \frac{\partial^2 U}{\partial x \partial z}, \frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 U}{\partial y \partial x} \right) = \underset{\sim}{0}.
\end{aligned}$$

[Note, we assume that the function U is sufficiently well-behaved for its partial second derivatives to commute.]

3.12 $\quad \operatorname{div}(\operatorname{curl} \underset{\sim}{F}) = 0$

Proof :

$$\operatorname{curl} \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \underset{\sim}{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \underset{\sim}{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \underset{\sim}{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

so

$$\operatorname{div}(\operatorname{curl} \underset{\sim}{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right).$$

$$= \frac{\partial^2 F_3}{\partial x\,\partial y} - \frac{\partial^2 F_2}{\partial x\,\partial z} + \frac{\partial^2 F_1}{\partial y\,\partial z} - \frac{\partial^2 F_3}{\partial y\,\partial x} + \frac{\partial^2 F_2}{\partial z\,\partial x} - \frac{\partial^2 F_1}{\partial z\,\partial y}$$

$$= 0 \qquad \left(\text{assuming } \frac{\partial^2 F_3}{\partial x\,\partial y} = \frac{\partial^2 F_3}{\partial y\,\partial x}\right).$$

3.13 $$\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{F}) = \underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{F}) - \nabla^2 \underset{\sim}{F}$$

or $$\text{curl}\,(\text{curl}\,\underset{\sim}{F}) = \text{grad}\,(\text{div}\,\underset{\sim}{F}) - \nabla^2 \underset{\sim}{F}.$$

3.14 $$\underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{F}) = \underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{F}) + \nabla^2 \underset{\sim}{F}$$

$$\text{grad}\,(\text{div}\,\underset{\sim}{F}) = \text{curl}\,(\text{curl}\,\underset{\sim}{F}) + \nabla^2 \underset{\sim}{F}.$$

These are just a rearrangement of each other, and involve the Laplacian of the vector field $\underset{\sim}{F}$, $\nabla^2 \underset{\sim}{F}$ which we met above.

$$\nabla^2 \underset{\sim}{F} = \text{grad}\,(\text{div}\,\underset{\sim}{F}) - \text{curl}\,(\text{curl}\,\underset{\sim}{F})$$

Of the above 3.10 – 3.14 ,

3.10 and 3.11 are especially important for the rest of the course.

3.10 defines the <u>Laplacian,</u> ∇^2.

The equation $\nabla^2 U = 0$ is called LAPLACE'S EQUATION, and we will solve some simple cases in Chapter 7.

3.11 is $\text{curl}(\text{grad } U) = \underset{\sim}{0}$.

It is interesting to ask, given a vector field $\underset{\sim}{F}$, can we find a scalar field U such that $\text{grad } U = \underset{\sim}{F}$?

If $\text{curl } \underset{\sim}{F} \neq \underset{\sim}{0}$, this is <u>not</u> possible :

define $\underset{\sim}{H} \equiv \text{grad } U$, 3.11 $\Rightarrow \text{curl } \underset{\sim}{H} = \underset{\sim}{0}$

$\Rightarrow \underset{\sim}{H} \neq \underset{\sim}{F}$ for any U.

In Chapter 4, we will show the converse :

if $\text{curl } \underset{\sim}{F} = \underset{\sim}{0}$, then there IS a U such that $\underset{\sim}{F} = \text{grad } U$.