Chapter 7

Laplace's Equation

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Syllabus section;
7. Laplace's equation. Uniqueness under suitable boundary conditions. Separation of variables. Two-dimensional solutions in Cartesian and polar coordinates. Axisymmetric spherical harmonic solutions.

7.1 The Laplace and Poisson equations

Let $\Phi(\mathbf{r})$ be a scalar field in three dimensions, as in previous chapters. Laplace's equation is simply

$$\nabla^2\Phi = 0 \tag{7.1}$$

where, as we met in Chapter 3.6, $\nabla^2\Phi \equiv \nabla\cdot(\nabla\Phi) \equiv \mathrm{div}(\mathrm{grad}\,\Phi)$; here ∇^2 is called the **Laplacian operator**, or just the Laplacian.

Remember from before, if Φ is a scalar field, its gradient $\nabla\Phi$ is a vector field, and then taking div of that gives us another scalar field: so Laplace's equation is a scalar equation.

In Cartesian x,y,z coordinates, things are simple: we recall the definitions from Chapter 3,

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\,\mathbf{i} + \frac{\partial\Phi}{\partial y}\,\mathbf{j} + \frac{\partial\Phi}{\partial z}\,\mathbf{k}$$

and

$$\nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Putting $\mathbf{F} = \nabla\Phi$ above, so $F_1 = \partial\Phi/\partial x$ etc, Laplace's equation in Cartesians is

$$\nabla^2\Phi \equiv \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{7.2}$$

Note that if we are using other coordinates (e.g. cylindrical polars or spherical polars) we must use results for grad and div in those coordinates from Chapter 5, so it will look different; we look at those later.

Laplace's equation often occurs as follows: suppose we have a conservative vector field $\mathbf{F}$, so that $\mathbf{F} = \nabla\Phi$ for some scalar field Φ as in Chapter 4.7; then if $\nabla\cdot\mathbf{F} = 0$ this gives Laplace's equation $\nabla^2\Phi = 0$.

Aside: Laplace's equation is the simplest and most basic example of one of the three types of second-order linear partial differential equations (PDEs), known as the *"elliptic"* type. Laplace's equation is a linear homogeneous equation.

A generalisation of Laplace's equation is **Poisson's equation** which is

$$\nabla^2\Phi = f(\mathbf{r}) \quad ,$$

where $f(\mathbf{r})$ is a given scalar field. Laplace's equation is clearly a special case of Poisson's where $f(\mathbf{r}) = 0$ at all points in the volume of interest.

The basic examples of the other types of PDE are the **wave equation**

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \nabla^2 f \ ,$$

where c is constant (usually the speed of sound or light) and t is time;
and the **heat equation** or **diffusion equation**

$$\frac{\partial f}{\partial t} = \kappa\nabla^2 f$$

where f is temperature in a solid, and κ is a constant. (We met the heat equation with a single spatial variable in Example 6.8 on Fourier series).

In maths, the wave equation is an example of a *"hyperbolic"* PDE and and the heat equation is a *"parabolic"* PDE. These names are potentially confusing since the solutions have nothing to do with ellipses, parabolas, or hyperbolas, but this is just a "shorthand" because the powers and signs in the equations look rather similar to the equations for ellipsoids, paraboloids and hyperboloids from Chapter 1.

Laplace's and Poisson's equations are very important, both because of their occurrence in many physics applications, and because they are the basic examples of elliptic PDEs. We are now going to spend the rest of this chapter considering some solutions of Laplace's equation in 2 dimensions.

We can see directly that there are some simple solutions of Laplace's equation, e.g.

$$\begin{aligned}
\Phi &= c \qquad \text{constant} \\
\Phi &= x \\
\Phi &= y \\
\Phi &= xy \\
\Phi &= x^2 - y^2 \\
& \quad \text{etc}
\end{aligned}$$

These clearly are solutions, by direct evaluation of $\nabla^2\Phi$ from Eq. 7.2. There are in fact an infinite number of general solutions to Laplace's equation, which are known as **harmonic functions**.

We easily see that ∇^2 is a linear operator: that is

$$\nabla^2(\lambda\Phi_1 + \mu\Phi_2) = \lambda\nabla^2\Phi_1 + \mu\nabla^2\Phi_2 \quad .$$

for any two scalar fields Φ_1, Φ_2 and any two constants λ, μ (independent of position), since both grad and div have this property. Hence if Φ_1 and Φ_2 are both solutions of Laplace's equation, so is $\lambda\Phi_1 + \mu\Phi_2$. Also, if Ψ is a solution of Poisson's equation and Φ is a solution of Laplace's equation, $\Psi + \Phi$ is also a solution of Poisson's equation, for the same $f(\mathbf{r})$.

Aside: Considering some gravitational and electromagnetic examples of conservative fields, and using the Divergence Theorem

$$\int_V \nabla\cdot\mathbf{F}\,\mathrm{d}V = \int_S \mathbf{F}.\mathrm{d}\mathbf{S}$$

we see that if $\nabla \cdot \mathbf{F} = 0$ everywhere there are no sources inside the volume, which for gravity means that there is no mass there, and for electric field means that there is no (net) charge. Hence, Laplace's equation describes the gravitational potential in regions of space where there is no matter, and the electric potential in regions where there are no charges.

If instead there is a net charge density ρ, the electric field $\mathbf{E}$ satisfies

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho(\mathbf{r})$$

where ε_0 is a constant of nature. (This is one of the four Maxwell's equations). Combining this with $\mathbf{E} = -\nabla\Phi$ gives

$$\nabla^2 \Phi = -\frac{1}{\varepsilon_0} \rho(\mathbf{r}).$$

That is an example of Poisson's equation as we met above. Laplace's equation is of course a special case of Poisson's equation, in which the function on the right-hand side is zero throughout the volume of interest.

7.2 Uniqueness of Solutions to Poisson's (and Laplace's) Equation

Here, we will prove that under suitable boundary conditions the solution of Poisson's (or Laplace's) equation is **unique**. We shall then investigate what the solutions actually are in some simple cases, in each of Cartesian, cylindrical and spherical polar coordinates.

As is common in differential equations, there are many general solutions (in fact an infinite family), so to find the solution in a specific case we need to be given some **boundary conditions**. Recall for a 1-D ordinary differential equation we often need a function value at one or two ends of a line; but here since Laplace's equation works in 3 dimensions, usually we need the value of $\Phi(\mathbf{r})$ to be given at all points on a closed surface S, and we solve Laplace's equation to find Φ in the volume **inside** S. (Occasionally we solve over the infinite volume outside S, with another boundary condition for Φ at infinity).

Theorem 7.1 *Suppose that $\nabla^2 U = f(\mathbf{r})$ throughout some closed volume V, $f(\mathbf{r})$ being some specified function of $\mathbf{r}$, and that the value of U is specified at every point on the surface S bounding volume V. Then, if a solution $U(\mathbf{r})$ exists to this problem, it is unique.*

Proof:
Before proceeding, we need to recall Eq. 3.6, which was

$$\nabla \cdot (U\mathbf{F}) = U\nabla \cdot \mathbf{F} + (\nabla U) \cdot \mathbf{F} \qquad .$$

Choosing $\mathbf{F} = \nabla U$ in the above, we get the identity

$$\nabla \cdot (U\nabla U) = U\nabla^2 U + (\nabla U) \cdot (\nabla U) \qquad (*)$$

which we use below.

Now to prove the uniqueness theorem, suppose that U_1 and U_2 are two scalar fields which *both* solve the given problem. Define $W \equiv U_1 - U_2$ to be the difference of our two solutions.

Then, we know that $\nabla^2 W = 0$ inside volume V (by linearity), and W = 0 at all points on the surface *S*, since both U_1 and U_2 match the given boundary condition.

Now we consider the volume integral

$$\begin{aligned}
\int_V |\nabla W|^2 \, \mathrm{d}V &= \int_V (\nabla W).(\nabla W) \, \mathrm{d}V \\
&= \int_V \nabla \cdot (W \nabla W) - W \nabla^2 W \ \mathrm{d}V \quad \text{ using } (*) \text{ above} \\
&= \int_V \nabla \cdot (W \nabla W) \ dV \ - 0 \qquad \text{since } \nabla^2 W = 0 \text{ everywhere in } V \\
&= \int_S (W \nabla W) \cdot \mathrm{d}\mathbf{S} \qquad \text{(by the Divergence Theorem)} \\
&= 0 \qquad \text{because } W = 0 \text{ on } S
\end{aligned}$$

Now, the integrand on the LHS is a squared quantity, therefore is always non-negative, and its integral is zero. This can only happen if $\nabla W = \mathbf{0}$ throughout V (otherwise, if ∇W was non-zero anywhere in V, the whole integral on the LHS will be positive because there cannot be any negative bits in the integrand to cancel the positive part, i.e. a contradiction).

Now $\nabla W = \mathbf{0}$ throughout V means W is a constant throughout V. But $W = 0$ on the boundary of V, therefore $W = 0$ throughout V. Hence $U_1 = U_2$ throughout V, so the solution is **unique**. Q.E.D.

Note that we have actually proved uniqueness for Poisson's equation, and Laplace's is a special case of that.

[**Aside:** It is fairly clear that the final step in the displayed calculation above also works if, instead of $W = 0$ on the boundary, $\nabla W \cdot \mathbf{n} = 0$ where $\mathbf{n}$ is the normal to the surface S. This corresponds to being given a boundary condition for $\nabla U \cdot \mathbf{n}$ on the boundary, instead of the value of U itself. Moreover, it still works if at each point on the boundary either U or $\nabla U \cdot \mathbf{n}$ is specified. The case where U is given on the boundary is called "*Dirichlet boundary conditions*", and the case where $\nabla U \cdot \mathbf{n}$ is given is called "*Neumann boundary conditions*". If we only have Neumann conditions, our W above is still a constant but not necessarily zero, so the solution U is unique up to addition of any arbitrary constant. We will only deal with Dirichlet boundary conditions from here on, but you may meet the Neumann conditions in later courses.]

The virtue of this uniqueness theorem is that it gives us a licence to make whatever assumptions or guesses we like, provided we can justify them afterwards by showing both Laplace's equation and the boundary conditions are satisfied: if they are, the solution we found must be the right one, even if our method involved some educated guesses.

Having proved uniqueness, we now demonstrate how to actually find solutions of Laplace's equation in some simple situations. In general $\Phi(\mathbf{r})$ can depend on all three coordinates, but we will confine ourselves to cases depending on two of the three coordinates: we will study the three most common coordinate systems as before:

- In Cartesian coordinates, we will take $\Phi(x,y)$, so Φ does not depend on z.
- In cylindrical polar coordinates, we will take $U(\rho,\phi)$ so U does not depend on z again, and we relabel Φ to U to avoid confusion with the angle ϕ.
- In spherical polar coordinates, we will take $U(r,\theta)$, so U does not depend on ϕ and we have rotational symmetry around the z axis.

The first two of these cases provide us with a nice geometrical interpretation. For $\Phi(x,y)$ or $U(\rho,\phi)$, we can forget about the $z-$ coordinate: then things reduce to a two dimensional problem, and we have boundary conditions given on the edge(s) of a region, (say a rectangle or circle) and we have to solve for Φ or U **inside** the given region. Now imagine Φ as a varying height h. Solving Laplace's equation in 2D subject to boundary conditions is like taking a rubber sheet with its edges stuck to a rigid frame at the boundary with a "warp" in the third dimension: the frame fixes the height at the boundary, while the rubber tries to minimize its total area, which is equivalent to solving Laplace's equation.

For spherical polars $U(r,\theta)$ though, the 2-D interpretation no longer applies because the sphere still lives in 3-D.

Note: A "physical" example in three dimensions is as follows: suppose we take a uniform solid object (of arbitrary shape), and attach a large number of tiny thermostat-controlled heater/coolers to the surface, and set all the thermostats to some smoothly-varying function on the surface. The temperature inside, $T(\mathbf{r})$, will obey the heat equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T \quad ,$$

with κ a constant and boundary conditions set by our thermostats. If we wait a long enough time so the temperature distribution inside converges to a steady state, the LHS above will then be zero, so then the temperature inside the solid will solve Laplace's equation, with the given surface settings as the boundary condition. (If our boundary condition is $T =$constant independent of position, we just get the obvious boring solution $T =$ constant inside; but if the boundary settings vary around the surface, it becomes an interesting problem.)

The choice of coordinates will be adapted to the geometry of the domain of interest and its boundaries, which usually makes calculations easier. For rectangular boundaries we use Cartesians, for circles or cylinders we use cylindrical polars, and for spherical boundaries we use spherical polars. For example, one may need to calculate the electrostatic potential outside a charged sphere. This would be very messy in Cartesian coordinates, and is much simpler if we use spherical polar coordinates instead. (This was one of the main reasons for studying Chapter 5)

7.3 2-D solutions of Laplace's equation in Cartesian coordinates

We first develop a general method for finding solutions $\Phi = \Phi(x,y)$ to Laplace's equation inside a rectangular domain, with given boundary conditions for Φ on all four edges of the rectangle. In Cartesian coordinates, as we saw above, Laplace's equation is

$$\nabla^2\Phi = \nabla \cdot (\nabla\Phi) = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \quad . \tag{7.3}$$

and in two dimensions we just drop the last term.

We will now try looking for a solution of the form

$$\Phi(x,y) = X(x)Y(y).$$

where $X(x)$ is some function of x only, and $Y(y)$ is some function of y only. Such a solution is called a **separable solution**. We cannot justify this in advance, but if it works then the uniqueness theorem tells us we are OK. It is possible to prove that any solution can be written as a sum (possibly an infinite sum) of separable solutions, but this is beyond the scope of this course.

Substituting the above Φ into (7.3) gives

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2}Y + X\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} = 0.$$

Dividing this by XY gives

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = -\frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}.$$

Now, the left-hand side is a function of x **only**, and the right-hand side is a function of y **only**. This can only be satisfied if both sides are the same unknown constant.

Note: to prove the constant, $X(x)$ and $Y(y)$ must satisfy the above at any x,y inside our rectangle: so consider the above equation along a line (x_0,y) with fixed $x=x_0$ and varying y. The LHS is fixed, so the RHS must therefore be independent of y, i.e. constant. The same argument with y_0 fixed and x varying shows the LHS is constant, and it must be the same constant.

Now we call that constant $-\lambda$ with the minus sign for convenience, and both sides above equal $-\lambda$. Thus we have

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2}+\lambda X=0 \quad \text{and} \quad \frac{\mathrm{d}^2Y}{\mathrm{d}y^2}-\lambda Y=0\ .$$

If $\lambda \neq 0$, these equations are the differential equations for trigonometric and hyperbolic functions, which we met in chapter 1, so we know their general solutions as follows:

If λ is positive, define $k=\sqrt{\lambda}$ and the solution is

$$X=A\cos kx+B\sin kx, \quad Y=C\cosh ky+D\sinh ky,$$

where A,B,C,D are any constants. Multiplying these together,

$$\Phi=(A\cos kx+B\sin kx)(C\cosh ky+D\sinh ky)\ . \tag{7.4}$$

If λ is negative, define $k=\sqrt{-\lambda}$ and then the solution is

$$X=\hat{A}\cosh kx+\hat{B}\sinh kx, \quad Y=\hat{C}\cos ky+\hat{D}\sin ky.$$

where $\hat{A},\hat{B},\hat{C},\hat{D}$ are different constants. Then

$$\Phi=\left(\hat{A}\cosh kx+\hat{B}\sinh kx\right)\left(\hat{C}\cos ky+\hat{D}\sin ky\right)\ . \tag{7.5}$$

Note: in each of these solutions there is usually one more constant than we really need. For example if in (7.4) $AC\neq 0$ we can write

$$\Phi=AC\,(\cos kx+B/A\sin kx)\,(\cosh ky+D/C\sinh ky)$$

using just three constants AC, B/A and D/C: this means that in examples, one of the four constants can usually be set to 1. One way to do this is to write (7.4) as

$$\Phi=L\sin(kx+M)\sinh(ky+N)$$

for some constants L, M, and N. Usually this works fine, except it does not cover the case where $D=0$.

Finally, we need to deal separately with the case $\lambda=0$:
that easily gives us solutions $X=A_0x+B_0$ and $Y=C_0y+D_0$ so

$$\Phi=(A_0x+B_0)(C_0y+D_0)\ ,$$

with more constants A_0, B_0, C_0 and D_0. It is usually convenient to multiply this out and re-write it as

$$\Phi=\alpha+\beta x+\gamma y+\delta xy \tag{7.6}$$

with $\alpha,\beta,\gamma,\delta$ as alternative constants.

Remember, from linearity, **any sum** of any of the above functions with any k and any constants is also a solution of Laplace's equation. So, if we are given a boundary condition, solving Laplace's equation basically reduces to choosing a **"pick-and-mix"** of any sum of the general solutions in order to satisfy all the given boundary conditions: if we manage to do that, then we have solved the problem (and our solution is unique). If we are lucky, a particular one of the separable solutions will do this, as we see in the next example.

Example 7.1. Find the solution of

$$\nabla^2\Phi \equiv \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \qquad (*)$$

inside the rectangle D: $0 \le x \le a$, $0 \le y \le b$, given boundary conditions $\Phi = 0$ on the three sides $x = 0$, $y = 0$ and $x = a$; and $\Phi = \sin(p\pi x/a)$ on $y = b$, for some integer p.

We note here that Φ is zero along three of the sides, and non-zero along the "top" side with $y = b$. Also since $\sin 0 = 0$ and $\sin(p\pi a/a) = 0$, Φ is zero at the points $(0,b)$ and (a,b) so the given boundary condition is continuous at the corners.

Can we satisfy the boundary conditions in this case with one of the separable solutions above ? We consider them one by one. Clearly (7.6) will not work since it doesn't contain a sin. The form (7.4) is more promising, since if we take that equation and choose

$$A = 0 \qquad B = 1 \qquad k = \frac{p\pi}{a}$$

in there, the first bracket becomes $1\sin(p\pi x/a)$ which is the function we want on the boundary $y = b$. Now we just need to choose C, D to make the second bracket in 7.4 equal zero on the side $y = 0$, and 1 on the side $y = b$; this gives us two simultaneous equations for C, D :

$$C\cosh 0 + D\sinh 0 = 0$$

$$C\cosh(n\pi b/a) + D\sinh(n\pi b/a) = 1 \ ,$$

and the first of these implies $C = 0$, then the second gives $D = 1/\sinh(n\pi b/a)$.

Finally putting the above A, B, C, D back into 7.4 gives us

$$\Phi(x,y) = \sin\frac{p\pi x}{a}\sinh\frac{p\pi y}{a}\Big/\sinh\frac{p\pi b}{a} \quad ;$$

this satisfies all the boundary conditions and Laplace's equation, so it is the unique solution.

In the above Example, we chose a "sin" in the boundary condition to make it easy: but for more general boundary conditions, using just one separable solution will not work.

However, since Laplace's equation is linear, we can add together separable solutions to get a more general solution. In many cases, including the Cartesian one, it is possible to prove that every solution can be written as a sum of separable solutions (this is called completeness of the separable solutions).

In the Cartesian case we would need to introduce different values of A for each k etc., which we typically would denote A_k. Since k can take any value, the "sum" of separable solutions can in general become an integral[1] over k; but for the rectangular boundaries in the example above we will only need to take integer values of p, call it n, so the general solution of Laplace's equation inside the rectangle becomes

$$\begin{aligned}\Phi(x,y) \quad = \quad & \alpha + \beta x + \gamma y + \delta xy \\ & + \sum_{n=1}^{\infty}(A_n\cos n\pi x/a + B_n\sin n\pi x/a)(C_n\cosh n\pi y/a + D_n\sinh n\pi y/a) \\ & + \sum_{n=1}^{\infty}(a_n\cosh n\pi x/b + b_n\sinh n\pi x/b)(c_n\cos n\pi y/b + d_n\sin n\pi y/b)\end{aligned} \qquad (7.7)$$

[1]This leads to the use of Fourier transforms, which is the next step, beyond this course, in Fourier methods

We note that the $\sin n\pi x/a$ terms vanish at $x=0$ and $x=a$ so they will fit Dirichlet boundary conditions which are zero on those boundaries. If multiplied by a $\sinh n\pi y$ they also vanish on $y=0$ so are non-zero only on $y=b$: to get similar forms which are zero at $y=b$ and non-zero at $y=0$ we need to take a combination of $\sinh n\pi y$ and $\cosh n\pi y$ which is zero at $y=b$: using the addition formula, this will turn out to look like $\sinh n\pi(b-y)/a$.

(The $\cos n\pi x/a$ terms are not zero on the boundary, but have vanishing derivative $\mathbf{n}.\nabla\Phi = \partial\Phi/\partial x$ at $x=0$ and $x=a$, so they will fit Neumann boundary conditions which are zero on those boundaries. Since we will stick to Dirichlet problems as examples in this course, we will find we are using only the sine terms not cos terms in our solutions).

For the other two sides at $x=0$ and $x=a$, we just repeat the above interchanging $x \leftrightarrow y$ and $a \leftrightarrow b$: so a solution which is non-zero only on side $x=a$ will look like $\sin n\pi y/b \sinh n\pi x/b$, and a solution which is non-zero only on the side $x=0$ will look like $\sin n\pi y/b \sinh n\pi(a-x)/b$.

From these remarks, we can see that in order to fit general boundary conditions, we can solve it if we break our function on the boundary into a (possibly infinite) sum of sin/cos functions i.e. a Fourier series.

Now we look at a boundary condition with a general function on one side:

Example 7.2. Consider the previous example but with $\Phi = g(x)$ on side $y=b$ for some given $g(x)$, and $\Phi=0$ on the other three sides of the rectangle.

We try a linear combination of solutions of the form found above (keeping the conditions derived from the other parts of the boundary):

$$\Phi(x,y) = \sum_{n=1}^{\infty} D_n \sinh\frac{n\pi y}{a}\sin\frac{n\pi x}{a}.$$

Each term on the RHS is automatically a solution of Laplace's equation and is zero on the other three sides, so we just need to choose a set of constants D_n's to make this match the given $g(x)$ along the side $y=b$.

Putting in $y=b$ above gives us

$$\Phi(x,b) = \sum_{n=1}^{\infty} D_n \sinh\frac{n\pi b}{a}\sin\frac{n\pi x}{a} \qquad = g(x).$$

here the D_n and the sinh don't depend on x so we can rewrite this as

$$\Phi(x,b) = \sum_{1}^{\infty} E_n \sin\frac{n\pi x}{a} \qquad = g(x) \tag{$*$}$$

with $E_n \equiv D_n \sinh(n\pi b/a)$.

Finding the coefficients E_n in equation $(*)$ is a standard problem in (arbitrary range) Fourier series from the previous Chapter. The answer is

$$E_n = \frac{2}{a}\int_0^a g(x)\sin\frac{n\pi x}{a}\,\mathrm{d}x.$$

Now we just need to evaluate this integral for all n, and then plug in $D_n = E_n/\sinh(n\pi b/a)$ back to the original equation to give us a solution

$$\Phi(x,y) = \sum_{n=1}^{\infty} \frac{E_n}{\sinh(n\pi b/a)}\sinh\frac{n\pi y}{a}\sin\frac{n\pi x}{a}.$$

By uniqueness, we have found *the* solution.

We still have a couple more issues to deal with. So far, we have seen how to solve the problem as a Fourier series when the boundary conditions are zero on three sides and non-zero on any one side.

If the boundary conditions are non-zero on all four sides but still zero at all four corners, we can solve this just by breaking it into four separate problems, each of which has non-zero boundary values on exactly one side: this gives four solutions $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ each solving one different side: then add the four solutions, by linearity of Laplace's equation.

If the four corners are all one constant value, just subtract this constant from the boundary conditions, solve as above, and add the constant back to the final solution.

Finally, we have to deal with the case where the given boundary conditions are different (but still continuous) at the four corners. This can be dealt with by Eq. 7.6 above: given the four values at the corners, it is straightforward to choose our four constants $\alpha, \beta, \gamma, \delta$ to give a solution (call it Φ_0) which matches the given boundary values at all four corners, by starting with the $(0,0)$ corner, then the $(0,a)$, etc. Next, we subtract that $\Phi_0(x,y)$ from **all** the given boundary conditions on the edges to get a new set of boundary conditions for $\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4$; solve Φ_1 to Φ_4 by treating the four sides separately as above: and finally add all five solutions $\Phi_0 + \ldots + \Phi_4$ to get the answer.

This whole process is quite lengthy, but we have seen how to do it in principle.

Example 7.3. Consider a rectangle with $0 \le x \le 2$, $0 \le y \le 1$, and boundary values for $\Phi(x,0) = \sin \pi x$ etc. as shown at the left diagram in Figure 7.1.

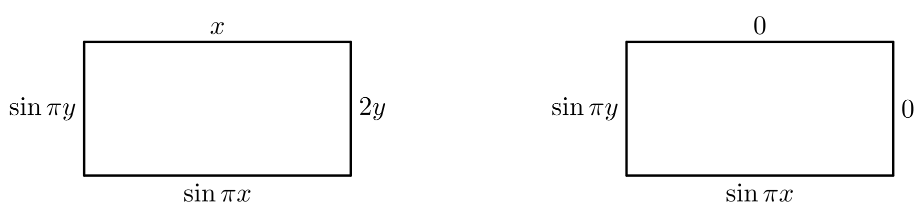

Figure 7.1: Left: boundary conditions on $\Phi(x,y)$. Right: boundary conditions after subtracting off $\Phi_0 = xy$ along the edges.

First we look at the boundary values at the four corners: reading these off the diagram we have $\Phi = 0$ at three corners and $\Phi(2,1) = 2$ at the corner $(x = 2, y = 1)$.
So, now we solve for the coefficients in $\Phi_0(x,y) = \alpha + \beta x + \gamma y + \delta xy$ so as to fit the given boundary values only at the four corners: starting at the origin and working out is easiest, so
$\Phi_0(0,0) = 0 \Rightarrow \alpha = 0$,
$\Phi_0(2,0) = 0 \Rightarrow \beta = 0$,
$\Phi_0(0,1) = 0 \Rightarrow \gamma = 0$,
$\Phi_0(2,1) = 2 \Rightarrow \delta = 1$;
therefore

$$\Phi_0(x,y) = 0 + 0 + 0 + 1xy \quad = xy$$

Now we evaluate Φ_0 along all four edges: it is zero on the left and bottom edges, it is $\Phi_0(2,y) = 2y$ on the right edge and $\Phi_0(x,1) = x$ along the top edge. Subtracting those from the original boundary conditions leaves the new boundary conditions in the right panel of Figure 7.1: by construction, these are zero at all corners. We can now match $\Phi(x,0)$ along the bottom side using

$$\Phi_1(x,y) = \frac{\sinh(\pi(1-y)) \sin \pi x}{\sinh(\pi)}$$

(this is like example 7.1), and match $\Phi(0,y)$ along the left-hand side with

$$\Phi_2(x,y) = \frac{\sinh(\pi(2-x))\sin \pi y}{\sinh(2\pi)} \ .$$

The full solution is

$$\Phi(x,y) = \Phi_0 + \Phi_1 + \Phi_2 \ .$$

Exercise 7.1. Find $\Phi(x,y)$ in $0 < x < \pi$, $0 < y < 1$, satisfying the following conditions:

$$\nabla^2\Phi \ = 0 \quad \text{in } 0 < x < \pi, \ \ 0 < y < 1,$$

$$\Phi = \sin x \text{ on } y = 0$$

and $\Phi = 0$ on the other three sides of the rectangle. Is the solution unique? □

7.4 2-D solutions of Laplace's equation in cylindrical polar coordinates

We now look at cylindrical polar coordinates: this is the natural choice where the boundary conditions are given on a circle in 2D or a cylinder in 3D. It will turn out a bit simpler than Cartesians, since there are no corners to worry about on the boundary.

We also change our label for our scalar field from Φ to U, to avoid confusion with the angle ϕ (of course, this is just a re-labelling and makes no real difference).

From chapter 5, in cylindrical polar coordinates (ρ,ϕ,z), the grad of a scalar field U is

$$\nabla U = \frac{\partial U}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\mathbf{e}_\phi + \frac{\partial U}{\partial z}\mathbf{e}_z$$

and the divergence of $\mathbf{F} = F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z$ is

$$\nabla\cdot\mathbf{F} = \frac{1}{\rho}\left[\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{\partial F_\phi}{\partial \phi} + \frac{\partial(\rho F_z)}{\partial z}\right].$$

Putting these together we obtain

$$\nabla^2 U \ \equiv \ \mathrm{div}(\nabla U) \ = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial U}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial U}{\partial z}\right)\right] \ ,$$

which simplifies to

$$\nabla^2 U = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} \ .$$

Consider the case when everything in the problem is independent of z, so $U = U(\rho,\phi)$. Once again we seek a **separable** solution, this time we will write it as

$$U(\rho,\phi) = R(\rho)S(\phi) \ .$$

where R and S are functions to be found. Putting this into $\nabla^2 U$, working out and dividing by RS gives

$$\frac{\rho}{R}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) = \frac{-1}{S}\frac{\mathrm{d}^2 S}{\mathrm{d}\phi^2} \qquad (*)$$

Once again, the LHS is a function of only ρ and the RHS is a function of only ϕ, so by the same argument as before, both sides are some (unknown) constant, call it λ.

Setting the RHS above to λ, the differential equation for $S(\phi)$ is then

$$\frac{\mathrm{d}^2 S}{\mathrm{d}\phi^2} + \lambda S = 0 \; ,$$

which we met before: if $\lambda > 0$, it has the general solution

$$S(\phi) = A\cos(\sqrt{\lambda}\phi) + B\sin(\sqrt{\lambda}\phi) \; .$$

If $\lambda < 0$ we would similarly have

$$S(\phi) = \hat{A}\cosh(\sqrt{-\lambda}\phi) + \hat{B}\sinh(\sqrt{-\lambda}\phi) \; .$$

But, in polar coordinates $S(\phi)$ must be periodic, i.e. the solution must be the same if we add 2π to ϕ, since any pair of values ϕ_0 and $\phi_0 + 2\pi$ represent the same point in space; the sinh and cosh solutions with $\lambda < 0$ cannot obey this, so are “forbidden” and we discard them. The sin and cos solutions will obey this periodic condition iff $\sqrt{\lambda}$ is an **integer**. Thus, the only allowed values of λ are $\lambda = m^2$ where m is a positive integer (without loss of generality) and we can write the solution for a particular integer m as

$$S(\phi) = A_m \cos m\phi + B_m \sin m\phi \; .$$

Now, going back to $R(\rho)$ and setting the LHS of (*) equal to $\lambda = m^2$ gives

$$\rho \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho \frac{\mathrm{d}R}{\mathrm{d}\rho}\right) = m^2 R.$$

We guess a power-law solution $R = C\rho^q$ for constants C, q; substituting and working through, that simplifies to

$$q^2 = m^2$$

so $q = \pm m$. This is two independent solutions for q, and each has its own constant, so we write

$$R(\rho) = C_m \rho^m + D_m \rho^{-m} \; ,$$

and again C_m, D_m are constants; finally multiplying out S and R, we have a solution for U of the form

$$U(\rho,\phi) = (A_m \cos m\phi + B_m \sin m\phi)\left(C_m \rho^m + D_m \rho^{-m}\right).$$

for any integer $m > 0$.

The case $\lambda = 0$ is again a special case: then we integrate twice giving $R = C_0 \ln\rho + D_0$, and $S = A_0\phi + B_0$. In most cases we set $A_0 = 0$ by requiring uniqueness on adding 2π to ϕ; (but note there are special cases where it is acceptable for U not to be unique, provided ∇U is unique. This happens in fluid dynamics, for example, where we are interested in the fluid velocity $\mathbf{v} = \nabla U$ rather than the potential U itself. In that case we require that ∇U be single valued, which allows us to use an A_0 term).

Combining the above, the general solution of Laplace’s equation in cylindrical polars is a linear combination of all these above for the $m = 0$ case and every positive $m \geq 1$: each of these m has its own constants, so we get

$$U(\rho,\phi) = (A_0\phi + B_0)(C_0 \ln\rho + D_0) + \sum_{m=1}^{\infty} (A_m \cos m\phi + B_m \sin m\phi)\left(C_m \rho^m + D_m \rho^{-m}\right). \tag{7.8}$$

Note that this form implies that boundary conditions like $g(R,\phi)$ given on a circle or cylinder of fixed $\rho = R$ leads to a Fourier series problem in ϕ (once the terms in $A_0\phi$ have been found). However, in many cases we need only a finite number of terms and can use intelligent guesswork (essentially, including only terms with the **same** values of m which appear in the boundary conditions) to find the required set of constants.

Also, note the presence of both positive and negative powers of ρ on the RHS: if we are solving a problem **inside** a circle with boundary condition given on the circle, we will require all D_m to be zero for $m \geq 1$ so that the solution is bounded at the origin $\rho = 0$. Alternatively, we can be given boundary conditions on a circle and asked for a solution **outside** the circle, requiring the solution to be well-behaved at large $\rho \to \infty$: then we must set all C_m to zero for $m \geq 1$, and use only D_m terms.

Finally, we may be asked to solve Laplace's equation in an **annulus** between two circles of given radii, with boundary conditions given on both the inner and outer circles; in that case we will need to keep both C_m and D_m terms, and we'll get a pair of simultaneous equations for each m to match the given boundary conditions on both circles.

Example 7.4. Solve Laplace's equation $\nabla^2 U(\rho,\phi) = 0$ outside the unit circle, with boundary conditions $U(1,\phi) = 2\sin^2\phi$ on the unit circle, and $U \sim \ln\rho$ at large ρ.

First look at the general solution 7.8. That does not contain a $\sin^2$, but since $2\sin^2\phi = 1 - \cos 2\phi$, the latter form does look like a sum of two terms in 7.8: a constant ($m = 0$) term plus a $\cos 2\phi$ term which looks like an $m = 2$ term; so we can (correctly) guess that the same is true of the solution, i.e. we choose all $A_m \ldots D_m$ coefficients with $m = 1$ and $m \geq 3$ to be zero, so the infinite sum becomes just one term with $m = 2$. We also set $B_2 = 0$ since our boundary condition only contains a $\cos 2\phi$ not a $\sin 2\phi$.

The large-ρ condition implies $A_0 = 0$, and also $C_2 = 0$ since we don't want a ρ^{+2} term at large ρ.

Writing out 7.8 without all those zeros leaves us with our "educated guess" solution as

$$U = B_0 D_0 + B_0 C_0 \ln\rho + A_2 D_2 \cos(2\phi)\,\rho^{-2} \ .$$

Again this has several "redundant" constants, and we can just rewrite it as

$$U = \alpha + \beta\ln\rho + \gamma\cos(2\phi)\,\rho^{-2}$$

Finally, matching the given boundary values on the circle $\rho = 1$ gives us $\alpha = 1, \gamma = -1$, and the large ρ condition gives us $\beta = 1$, so the unique solution is

$$U(\rho,\phi) = 1 + \ln\rho - \frac{\cos 2\phi}{\rho^2} \ .$$

We can check this easily: it is a particular case of 7.8 so it does solve Laplace's equation. And it matches the given boundary conditions on $\rho = 1$ and large ρ; so it is the unique solution.

Exercise 7.2. Consider the region D defined by $a \leq \rho \leq b$, $0 \leq \phi \leq \pi$, $-\infty < z < \infty$. Sketch the region in a plane perpendicular to the z-axis which lies in D. On the boundaries $\rho = a$, $\phi = 0$ and $\phi = \pi$, $U = 0$ while on the boundary $\rho = b$, $U = \phi\sin\phi$. Find the solution U of Laplace's equation in D, independent of z, which satisfies these boundary conditions.

[You may assume that on $0 \leq \phi \leq \pi$

$$\phi\sin\phi = \sum_{k=1}^{\infty} \frac{16k}{\pi(4k^2-1)^2} \sin 2k\phi.]$$

□

Note: Finally, it is also worth noting that solutions like $\rho^n\cos(n\phi)$ can also be expanded as polynomials in x, y: for example, $\cos(4\phi) = 8\cos^4\phi - 8\cos^2\phi + 1$, and $\rho^4 = (x^2+y^2)^2$, therefore a bit of arithmetic leads to $\rho^4\cos 4\phi \equiv x^4 - 6x^2y^2 + y^4$, and you can use the Cartesian formula to check that ∇^2 of that is zero. These may occasionally be useful, but they rapidly get unmanageable for large n.

7.5 Axisymmetric solutions of Laplace's equation in spherical polar coordinates

Now we consider what to do in problems with a naturally spherical geometry. First, we need to work out what $\nabla^2 U$ is in spherical polar coordinates.

As before, we have

$$\nabla^2 U \equiv \operatorname{div}(\nabla U).$$

which is true in any coordinate system. Now in spherical polar coordinates,

$$\nabla U = \frac{\partial U}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\mathbf{e}_\phi$$

and the divergence of $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi$ is

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2\sin\theta}\left[\frac{\partial(r^2\sin\theta F_r)}{\partial r} + \frac{\partial(r\sin\theta F_\theta)}{\partial\theta} + \frac{\partial(rF_\phi)}{\partial\phi}\right].$$

Putting these together we obtain

$$\nabla^2 U = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial U}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial U}{\partial\phi}\right)\right],$$

which simplifies to

$$\nabla^2 U = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}\right].$$

Many problems are *axisymmetric* – that is, there is no dependence on the ϕ coordinate. In such cases $U = U(r,\theta)$ and $\partial(\text{anything})/\partial\phi = 0$. As in the previous cases, we proceed by seeking a **separable** solution:

$$U(r,\theta) = R(r)S(\theta).$$

[different meanings from the R and S in the last section]. Thus $\nabla^2 U = 0$ becomes

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)S + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right)R\right] = 0$$

which rearranges to

$$\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = \frac{-1}{S(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right).$$

Once again, the left-hand side is a function of r only, and the right-hand side is a function of θ only. But they are equal, and so they must both be some constant, say λ. Thus

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \lambda R = 0 \tag{7.9}$$

and

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}S}{\mathrm{d}\theta}\right) + \lambda S = 0. \tag{7.10}$$

We consider equation (7.10) first. If we define $w = \cos\theta$, then

$$\frac{\mathrm{d}}{\mathrm{d}w} = \frac{-1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta},$$

so equation (7.10) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}w}\left((1-w^2)\frac{\mathrm{d}S}{\mathrm{d}w}\right)+\lambda S=0.$$

which is called **Legendre's differential equation**. We see in the next Section that only **Legendre polynomial** solutions $S=P_\ell(w)=P_\ell(\cos\theta)$ are allowed, i.e. the cases where $\lambda=\ell(\ell+1)$ and ℓ is an integer, and P_ℓ is the Legendre polynomial of order ℓ.

Going back to equation (7.9), inserting $\lambda=\ell(\ell+1)$ then $R(r)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)-\ell(\ell+1)R=0. \tag{7.11}$$

We try looking for a power-law solution, $R=Ar^p$ of this with A,p constant: inserting it we find

$$p(p+1)Ar^p=\ell(\ell+1)Ar^p$$

i.e. $p(p+1)=\ell(\ell+1)$. Given ℓ, this is a quadratic equation for p. It has solutions $p=\ell$ and $p=-(\ell+1)$. Hence the general solution for $R(r)$ is

$$R=Ar^\ell+\frac{B}{r^{\ell+1}}.$$

and so the solution for U is

$$U(r,\theta)=\left(Ar^\ell+\frac{B}{r^{\ell+1}}\right)P_\ell(\cos\theta).$$

Because ∇^2 is a linear operator, any linear combination of solutions is also a solution of Laplace's equation , so again the general solution is an infinite sum:

$$U(r,\theta)=\sum_{n=0}^{\infty}\left(A_nr^n+\frac{B_n}{r^{n+1}}\right)P_n(\cos\theta). \tag{7.12}$$

The individual functions on the right are **axisymmetric spherical harmonics** and they form a set of axisymmetric solutions of Laplace's equation which is complete, i.e. (7.12) can be shown to be the most general axisymmetric solution.

One can match arbitrary boundary conditions to an infinite series of Legendre polynomials using their orthogonality properties (see later). However, in this course we will stick to problems where only a few terms are needed and we can see what they are by intelligent guesswork: the essential rule is **only** to put into the prospective answer those Legendre polynomials which appear in the boundary conditions.

Example 7.5. A perfectly spherical conductor, centre 0, radius a, is placed in an otherwise uniform electric field $\mathbf{E}_0$. (Mathematically, the condition for a conductor is that the electrostatic potential U is constant.) What is the potential everywhere outside the conductor? And inside?

Outside the conductor ($r>a$), we want to solve $\nabla^2U=0$. The boundary conditions are that $U=$constant on $r=a$ and that far from the conductor $\nabla U\to\mathbf{E}_0$.

The unperturbed field (the one before the conductor was added) is $\mathbf{E}=E_0\mathbf{k}$, choosing the z-axis to align with the field. Converting this to the $\mathbf{e}$'s of spherical polars, we have

$$\mathbf{E}_0=E_0\cos\theta\,\mathbf{e}_r-E_0\sin\theta\,\mathbf{e}_\theta$$

which is what the field must look like as $r\to\infty$: this has potential

$$U_0=E_0r\cos\theta+\text{ constant}=E_0rP_1(\cos\theta)+\text{ constant}.$$

(Note that this *is* a solution of Laplace's equation.) Now our potential

$$U = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta) \;\to\; \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

as $r\to\infty$. But this must equal $U_0 = E_0 r P_1(\cos\theta)$+const. at large r, so we can deduce that $A_1 = E_0$, A_0 is an arbitrary constant, and $A_n = 0$ for all other n.

On $r = a$ we want U to be constant, i.e. it should not vary with θ. Now on $r = a$

$$U(a,\theta) = A_0 + \frac{B_0}{a} + \left(E_0 a + \frac{B_1}{a^2}\right)P_1(\cos\theta) + \sum_{n=2}^{\infty}\frac{B_n}{a^{n+1}}P_n(\cos\theta)$$

The potential on $r = a$ *will* vary with θ unless all the coefficients of $P_n(\cos\theta)$ $(n>0)$ each vanish. Hence we must have $B_1 = -E_0 a^3$ to make the bracket vanish, and $B_n = 0$ $(n\geq 2)$. Hence finally the solution is

$$U(r,\theta) = A_0 + \frac{B_0}{r} + E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta.$$

Note that A_0 and B_0 are undetermined constants. To determine B_0 we need additional information to ascertain the potential difference between the surface of the conductor and a point at infinity. The constant A_0 will always be arbitrary, because the absolute value of the potential has no physical meaning (only its gradient is actually observable).

Inside, since U is constant on the boundary, it must be constant inside the conductor.

This last point has practical consequences. The voltage in space [in a static field] satisfies Laplace's equation. If you stand under an electricity pylon, there is a rather large voltage change—thousands of volts—between your head and your feet. But if you stand inside a wire cage (often called a Faraday cage), then the wire acts like a continuous conductor and equalizes the voltage over the cage and hence inside the cage too. That is why a wire cage provides a refuge from lightning. Cages also provide screening from electronic surveillance, or, by putting equipment inside them, safety for the people outside.

Exercise 7.3. Show that at a general point the following are solutions of Laplace's equation $\nabla^2 U = 0$.

1. $U = r^n\cos n\theta$, for an integer n, in cylindrical polar coordinates.
2. $U = r\sin\theta\cos\phi$, in spherical polar coordinates.

□

7.6 Introduction to Legendre polynomials

We now take a brief look at the Legendre polynomials. These are defined as the solutions of **Legendre's differential equation** which is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}f}{\mathrm{d}x}\right) + \lambda f = 0.$$

or similar, where λ is an arbitrary constant. The solution of this is outside the scope of this course, but essentially we search for power-law solutions of the form

$$f(x) = \sum a_p x^p \quad .$$

Then, it can be shown that the series only converges at both $x = \pm 1$ if $\lambda = \ell(\ell+1)$ where ℓ is an integer, and we can take ℓ as a non-negative integer without loss of generality.

Then, the function $f(x)$ which satisfies the above D.E. for $\lambda = \ell(\ell+1)$ is called the **Legendre polynomial** of "degree" ℓ, usually written $P_\ell(x)$. (It is common to use letter ℓ for this integer, since when things are extended to 3-D **spherical harmonics**, letters n and m are generally used for other functions in the r and ϕ coordinates.)

There is an arbitrary multiplicative constant in each P_ℓ, which is chosen so that $P_\ell(1) = 1$ for all ℓ. It turns out that P_ℓ is an ℓ-th order polynomial, and involves only even/odd powers of w if ℓ is even/odd.

The solutions can be obtained by **Rodrigues' formula**

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}[(x^2-1)^\ell] \tag{7.13}$$

There is also a recurrence relation between them,

$$P_{\ell+1}(x) = \frac{1}{\ell+1}\left[(2\ell+1)\,xP_\ell(x) - \ell P_{\ell-1}(x)\right]$$

which gives all of them, working upwards from P_0 and P_1.

Starting from Rodrigues's formula

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \end{aligned}$$

then the recurrence relation gives subsequent ones as

$$\begin{aligned} P_2(x) &= \frac{1}{2}(3x^2-1) \\ P_3(x) &= \frac{1}{2}(5x^3-3x) \\ P_4(x) &= \frac{1}{8}(35x^4-30x^2+3) \\ &\quad \text{etc} \end{aligned}$$

Another important property is **orthogonality**, i.e. the fact that

$$\begin{aligned} \int_{-1}^{1} P_m(w)P_n(w)\,dw &= 0 \quad \text{if } m \neq n \\ &= \frac{2}{2n+1} \quad \text{if } m = n \end{aligned}$$

This property enables us to express any general function as an infinite series of Legendre polynomials, by a device similar to that for calculating Fourier coefficients.

In this course we will only look at simple functions, in which case a general n-th order polynomial can be rearranged into a sum of the first n Legendre polynomials, e.g. suppose we are given a boundary condition in Laplace's equation looking like $f(w) = w^2 + w + 1$, ($w \equiv \cos\theta$) we need to choose a sum of Legendre polynomials to match this. We need $(2/3)P_2(w)$ to match the quadratic w^2 term. Then we need $1P_1(w)$ to match the linear term. Finally for the constant, the $(2/3)P_2$ has given us a $-1/3$ constant term, so to match the 1 we need $+(4/3)P_0$ on the right hand side. So in that example,

$$w^2 + w + 1 \equiv \frac{2}{3}P_2(w) + 1P_1(w) + \frac{4}{3}P_0(w)$$

and we can now put the right-hand-side into the general solution to Laplace's equation, 7.12 , and choose suitable constants to match.