Chapter 6

Fourier series

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Syllabus section:
6. Fourier series: full, half and arbitrary range series. Parseval's Theorem.

Fourier series provide a way to do various calculations with, and to analyse the behaviour of, functions which are **periodic**: this means that they repeat the same values in a regular pattern, or are defined in a finite range. Specifically, a function which is **periodic** with period L will obey an equation

$$f(x+L) = f(x) \qquad \text{for all } x$$

and to start with, we will assume $L = 2\pi$ also for convenience. We already know that $\cos nx$ and $\sin nx$ for any integer n have period 2π. (So, of course, do the other trigonometric functions such as $\tan x$, but these have the disadvantage of becoming unbounded at certain values, e.g. $\tan x$ is unbounded at $x = \pi/2$).

The basic principle of Fourier series is to express our periodic function $f(x)$ as an infinite sum of sine and cosine functions,

$$f(x) = \sum_0^\infty (a_n \cos nx + b_n \sin nx)$$

for a periodic and piecewise differentiable $f(x)$ (in fact, for any function defined on a range of length 2π). We will slightly modify this way of writing the series soon.

Such a series splits f into pieces of different "frequency": geometrically, each of the $\sin nx$ and *cosnx* terms has exactly n positive and negative"wiggles" over the range $0 \leq x \leq 2\pi$, and the a_n, b_n are constants telling us how much f varies at each different frequency.

This technique (and its generalisation to Fourier transforms) has a large number of practical applications, including: resolution of sound waves into their different frequencies, e.g. in MP3 players; telecommunications and Wi-Fi; computer graphics and image processing; astronomy and optics; climate variation; water waves; periodic behaviour of financial measures, etc.

6.1 Full range Fourier series

As above, the idea is that we have a given function $f(x)$ defined for a range of values of x of length 2π, say $-\pi \le x \le \pi$; now we approximate this function as an infinite sum of trigonometric functions, as

$$f(x) \approx S(x) \equiv \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \tag{6.1}$$

where a_n, b_n are an infinite series of constants to be determined. The right-hand side of this, $S(x)$ for short, is called the **Fourier series** for $f(x)$, and the set of coefficients a_n, b_n are called the **Fourier coefficients**. Here the $\frac{1}{2}a_0$ is really a $\cos 0x = 1$ constant term, and the $\frac{1}{2}$ is put in for convenience as we see below. (There is no point in including a b_0 term since $\sin 0x = 0$).

Clearly, to make progress we have to actually calculate the a_n, b_n; this looks very hard since we there are infinitely many of them, but is actually straightforward using the **orthogonality** properties of $\sin mx, \cos nx$: the key results we need are, for any two non-negative integers m and n,

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos mx \sin nx \, dx &= 0 \qquad (6.2)\\
\int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 2\pi & \text{if } m = n = 0 \end{cases}\\
\int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 0 & \text{if } m = n = 0 \end{cases}
\end{aligned}$$

All of the above are simple to prove using the trigonometric identities from Chapter 1, e.g. $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$ and similar. Using these, we can find the Fourier coefficients given $f(x)$: suppose we multiply Eq. 6.1 by $\cos mx$ for some *fixed* integer m, then integrate from $-\pi$ to π, then we have

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \int_{-\pi}^{\pi} \left[\tfrac{1}{2}a_0 \cos mx + \sum_{1}^{\infty} a_n \cos nx \cos mx + \sum_{1}^{\infty} b_n \sin nx \cos mx \right] dx$$

Assuming the sums converge, we can swap the integral sign and the summations above, giving

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \tfrac{1}{2}a_0 \left[\int_{-\pi}^{\pi} \cos mx \, dx\right] + \sum_{n=1}^{\infty} a_n \left[\int_{-\pi}^{\pi} \cos nx \cos mx \, dx\right] + \sum_{n=1}^{\infty} b_n \left[\int_{-\pi}^{\pi} \sin nx \cos mx \, dx\right] \tag{6.3}$$

Now suppose $m > 0$, and look at the integrals in square-brackets above: the first one is zero. From Eq. 6.2, the integrals in the middle term are all zero, except for exactly one case when $n = m$ when the integral is π. The integrals in the right-hand term are all zeros. Therefore, the RHS of the above is simply one non-zero term $= a_m \pi$; so rearranging we get

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, \mathrm{d}x \; .$$

Likewise, if instead we multiplied Eq. 6.1 by $\sin mx$ and integrated, we get

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \tfrac{1}{2}a_0 \left[\int_{-\pi}^{\pi} \sin mx \, dx\right] + \sum_{n=1}^{\infty} a_n \left[\int_{-\pi}^{\pi} \cos nx \sin mx \, dx\right] + \sum_{n=1}^{\infty} b_n \left[\int_{-\pi}^{\pi} \sin nx \sin mx \, dx\right]$$

Again all the square-brackets on the RHS are zero, except for one case in the rightmost bracket with $n = m$ which gives π; so the RHS is $b_m \pi$ and we rearrange to

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, \mathrm{d}x \; .$$

Finally, we need the special case of $m = 0$: going back to Eq. 6.3 the LHS contains $\cos mx = \cos 0x = 1$; now the a_0 term on the RHS is the only one which gives a non-zero integral, because both the infinite sums have $n \geq 1 \neq m$ and all the integrals are zero. Then the RHS above becomes $\frac{1}{2}a_0(2\pi)$, so the above equation for a_m is still correct for $m = 0$; note that the funny-looking $\frac{1}{2}$ in the original definition Eq. 6.1 was put in to make that work. (Some books may not have the $\frac{1}{2}$ in Eq. 6.1, but then we need to add a $\frac{1}{2}$ in the equation defining a_0 instead). Remember $\sin 0x = 0$ so there is no b_0 term to deal with.

The equations above were derived by choosing one **fixed** integer m and showing that all terms with $n \neq m$ disappeared: however the argument is correct for any value of m, so the above equations give **all** the coefficients a_m, b_m. (The choice of letter m above is arbitrary, but it had to be different to the n which runs from 0 to ∞). Finally, since m is a dummy label in the above and n has now disappeared, we can change the letter m back to n and we get

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \qquad (n \geq 0) \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \qquad (n \geq 1) \end{aligned} \tag{6.4}$$

Therefore, **to find the Fourier series** $S(x)$ for a given $f(x)$, we simply have to evaluate the definite integrals Eq. 6.4 (using a suitable method such as integration by parts) to get a_n, b_n for all n; then substitute those coefficients back into Eq. 6.1.

Next we take an example of actually evaluating the a_n, b_n for a given $f(x)$.

Example 6.1. Find the Fourier series for

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ x & \text{if } 0 < x < \pi. \end{cases}$$

Using the formulae above,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,\mathrm{d}x \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x \end{aligned}$$

(the lower limits become 0 because we were given $f(x) = 0$ in $[-\pi, 0]$, so that range contributes zero to the integrals). Evaluating the above, using integration by parts, we find that:

$$\begin{aligned} a_n &= \frac{1}{\pi}\left(\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,\mathrm{d}x\right) = \frac{1}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^{\pi} \\ &= \frac{1}{\pi n^2}(\cos n\pi - 1) \\ &= \frac{1}{\pi n^2}((-1)^n - 1) \end{aligned}$$

and this gives $a_n = -2/\pi n^2$ when n is odd, or $a_n = 0$ for even $n > 0$.

Note that for $n = 0$ the procedure above contains $0/0$ so is ill-defined: as is common, we need to treat $n = 0$ as a special case, with $\cos 0x = 1$:

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,1\,\mathrm{d}x = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}.$$

Finally we need the b_n's, which are

$$\begin{aligned} b_n &= \frac{1}{\pi}\left(\left[\frac{-x\cos nx}{n}\right]_0^\pi + \int_0^\pi \frac{\cos nx}{n}\,\mathrm{d}x\right) = \frac{1}{\pi}\left(-\frac{\pi\cos n\pi}{n} + \left[\frac{\sin nx}{n^2}\right]_0^\pi\right) \\ &= \frac{1}{\pi n}(-\pi\cos n\pi) + 0 = \frac{-(-1)^n}{n} \\ &= \frac{(-1)^{n+1}}{n} \end{aligned}$$

(and there is no b_0 term, so this gives b_n for all positive n).

Putting all these a_n, b_n back into the general form Eq. 6.1, the Fourier series we are asked for is

$$S(x) = \frac{\pi}{4} - \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)^2}\cos(2k+1)x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx\ .$$

where we have dealt with the odd/even n for a_n by replacing n with $2k+1$ which must be odd, and summing over $k = 0$ to ∞.

Although this general method always works (as long as we can evaluate the integrals), we do not need to do it for functions we can put into the required form by other means, as in the next example.

Example 6.2. Find the Fourier series for $\sin^4 x$.

Here we use the double angle formula: $\sin^4 x = \frac{1}{4}(1-\cos 2x)^2 = \frac{1}{4}(1-2\cos 2x + \cos^2 2x) = \frac{1}{4}(1 - 2\cos 2x + \frac{1}{2}[1+\cos 4x])$
so $\sin^4 x = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$.
This already looks like a special case of Eq. 6.1, so we just write $a_0 = \frac{3}{4}$ (remembering the half), $a_2 = -\frac{1}{2}$, $a_4 = \frac{1}{8}$; and all other a_n and all b_n are zero.

(**Note:** We could evaluate the integrals and get the same answer, but we don't need to do that here since we can see the result by inspection).

We note that the series $S(x)$ is periodic, i.e. if we take the same series for any x, rather than staying in the range $-\pi \le x \le \pi$, $S(x)$ will obey $S(x+2\pi) = S(x)$. So this can also be used for functions defined on a range longer than 2π if those functions are periodic with period 2π. Another way to look at this is that if we know the function on the range $[-\pi, \pi]$ we can define it for all x by insisting that it be periodic; graphically, this is equivalent to just "copying" the function infinitely many times for intervals 2π, like wallpaper.

We note that the range of x could equally well be $[\alpha, \alpha+2\pi]$ for any α, since all the quantities involved are periodic so this will give integrals over exactly the same range of values of f. Note $\alpha = 0$ is often used, so the range of x becomes $[0, 2\pi]$.

Exercise 6.1. Find the Fourier series of $f(x)$ defined by $f(x) = 0$ in $-\pi < x < 0$ and $f(x) = \cos x$ in $0 \le x < \pi$.
The answer should be

$$\tfrac{1}{2}\cos x + \sum_{k=1}^{\infty} \frac{4k}{\pi(4k^2-1)}\sin 2kx\ .$$

□

Going back to example 6.1, and evaluating both sides at $x = \pi/2$: we need to remember that the cosine of an odd multiple of $\pi/2$ is zero, the sine of an even multiple of $\pi/2$ is zero, and the sine of $(2k+1)\pi/2$ is

$(-1)^{k-1}$, from chapter 1. So we get

$$\begin{aligned} \frac{\pi}{2} &= \frac{\pi}{4} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \\ \Rightarrow \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \end{aligned}$$

A number of results of this sort, giving sums of numerical series, can be obtained by direct evaluation of equation (6.1) at some particular x. The only tricky point in using this is to guess which x to evaluate: usually one of π, $\pi/2$ or $\pi/4$ is what is needed, to make the sine and cosine functions give simple results such as 0 or $(-1)^n$ etc.

Warning: so far, we have not actually proved that the infinite sum $S(x)$ on the right-hand side of 6.1 actually converges, or has limit $f(x)$. Strictly, what we have shown is that IF there exists an infinite sum $S(x)$ which does converge to $f(x)$ over $-\pi \le x \le \pi$, then the coefficients must be given by Eq. 6.4.

We discuss the question of convergence and the limit in the next section.

6.2 Completeness and convergence of Fourier series

We now give answers to two questions: can every function with period 2π be written this way, and does the series $S(x)$ in 6.1 with coefficients 6.4 always converge at all x ? These ideas are referred to as completeness and convergence. To specify more fully, consider the sum of the first N terms with x fixed: this sum definitely exists since all the a_n, b_n are bounded if $f(x)$ is bounded, and we get a sum of a finite N bounded terms). Then let $N \to \infty$: if the limit exists, then $S(x)$ is said to converge at x. Completeness amounts to asking if this limit $S(x)$ equals the value of the original function $f(x)$. The proof of the relevant properties is not part of this course, but the result is. As usual, the conditions in it are like small print in contracts – ignorable most of the time, but important when things go wrong.

Theorem 6.1 *(Fourier's theorem or Dirichlet's theorem) If $f(x)$ is periodic with period 2π for all x, and $f(x)$ is piecewise smooth in $(-\pi,\pi)$, then the Fourier series $S(x)$ with coefficients a_n and b_n (defined as above) converges to $\frac{1}{2}(f(x+) + f(x-))$ at every point.*

Here "piecewise smooth" means sufficiently differentiable at all except isolated points, and $f(x+)$ means the limit of $f(x+\delta)$ as δ (positive) tends to zero, which is called the upper limit or right limit of $f(x)$ at x. Similarly $f(x-)$ is the limit of $f(x-\delta)$ as δ tends to zero, called the lower limit or left limit). At any x where $f(x)$ is continuous, we have $f(x+) = f(x-) = f(x)$, so $S(x) = \frac{1}{2}[f(x) + f(x)] = f(x)$ so the Fourier series does converge to exactly $f(x)$. At points where $f(x)$ has a discontinuity, $f(x+)$ and $f(x-)$ are not equal, and then $S(x) = \frac{1}{2}(f(x+) + f(x-))$ gives the average value of $f(x)$ on either side of the discontinuity: but this may not be the value of $f(x)$ itself at the point.

Typically, we will find that as $n \to \infty$, the coefficients a_n and b_n tend to zero like $1/n$ or faster.

Example 6.3. Taking the function and series of Example 6.1, Fourier's theorem tells us that at $x = \pi$ the series converges to $\frac{1}{2}(f(\pi+) + f(\pi-)) = \frac{1}{2}(0+\pi) = \frac{1}{2}\pi$, using $f(\pi+) = f((-\pi)+)$ by periodicity. The series then gives

$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)^2} \ ,$$

since $\sin n\pi = 0$ and $\cos(2k+1)\pi = -1$. Subtracting $\pi/4$ we have

$$\begin{aligned}\frac{\pi}{4} &= \sum_{k=1}^{\infty} \frac{2}{\pi(2k+1)^2} = \frac{2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}\ldots\right), \quad \text{therefore} \\ \frac{\pi^2}{8} &= 1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots \quad .\end{aligned}$$

As a nice corollary of the above, we can get the infinite sum for all integers (not just odd ones) as follows: define

$$T \equiv 1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots \qquad ,$$

then dividing by 4 gives

$$\frac{1}{4}T = \frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots$$

so subtracting,

$$\frac{3}{4}T = 1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

which is the series above. Therefore

$$T = \frac{4}{3}\frac{\pi^2}{8} = \frac{\pi^2}{6}$$

Note: There is a strange detail. Fourier's theorem tells us what happens in the limit of the infinite series. But if we take any finite number of terms we obviously cannot match a discontinuity exactly, since the finite series must give a continuous function. It turns out that any finite sum overshoots the function on either side of the discontinuity: this curious effect is called Gibbs's phenomenon— adding more terms does not reduce the overshoot, it just moves the overshoot closer to the discontinuity. (In the limit of the infinite sum, the overshoot gets "infinitesimally close" to the discontinuity, so for any x a finite distance from the discontinuity, this does not matter).

Example 6.4. The square wave.

Consider the "square wave" function defined by

$$f(x) = \begin{cases} 0 \text{ if } x<0 \\ 1 \text{ if } x>0 \end{cases} \tag{6.5}$$

in the domain $[-\pi,\pi]$ and periodic with period 2π. This gives

$$a_0 = 1 \qquad a_{n>0} = 0 \qquad b_n = \frac{1-\cos n\pi}{n\pi}$$

so b_n is 0 for even n or $2/(n\pi)$ for odd n. Therefore,

$$f(x) = \tfrac{1}{2} + 2\sum_{n \text{ odd}} \frac{\sin nx}{n\pi} \ . \tag{6.6}$$

Figure 6.1 shows the square wave and its approximations by its Fourier series (up to $n=1$ and $n=5$). Several things are noticeable:

(i) even a square wave, which looks very unlike sines and cosines, can be approximated by them, to any desired accuracy;

(ii) although we only considered the domain $[-\pi,\pi]$ the Fourier series automatically extends the domain to all real x by generating a periodic answer;

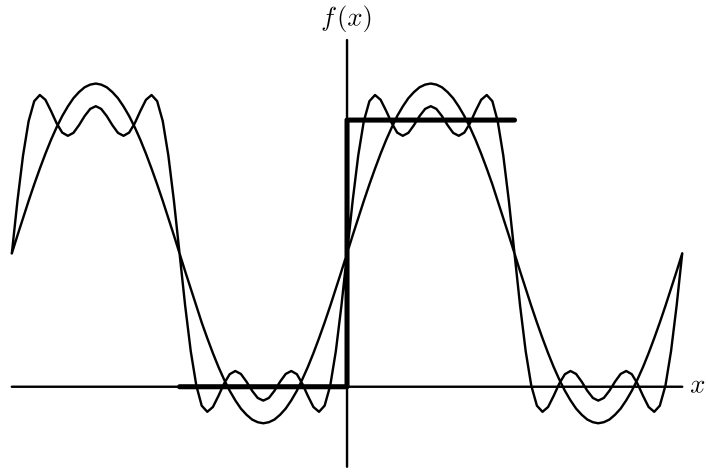

Figure 6.1: Square wave (as in equation (6.5) but with the vertical direction stretched for better visibility) and Fourier partial sums: two terms and four terms.

(iii) at discontinuities, the Fourier series gives the mean value of $f(x)$ on either side of the discontinuity.

(iv) close to discontinuities the Fourier series overshoots.

Another result telling us in what sense we have a good approximation is **Parseval's theorem:**

Theorem 6.2 *(Parseval's Theorem) If $f(x)$ has a Fourier series defined as in Section 6.1, then*

$$\int_{-\pi}^{\pi} f(x)^2 \, \mathrm{d}x = \tfrac{1}{2}\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

For a formal proof one has to deal with convergence of the infinite sum, but if we assume convergence we can write

$$f(x)^2 = \left(\tfrac{1}{2}a_0 + \sum_1^{\infty} a_n \cos nx + \sum_1^{\infty} b_n \sin nx\right)\left(\tfrac{1}{2}a_0 + \sum_1^{\infty} a_m \cos mx + \sum_1^{\infty} b_m \sin mx\right)$$

then we can expand this out into a double sum

$$\begin{aligned} f(x)^2 \quad &= \quad \frac{1}{4}a_0^2 + \tfrac{1}{2}a_0\left(\sum_1^{\infty} a_m \cos mx + \sum_1^{\infty} b_m \sin mx\right) + \tfrac{1}{2}a_0\left(\sum_1^{\infty} a_n \cos nx + \sum_1^{\infty} b_n \sin nx\right) \\ & \quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(a_n a_m \cos nx \cos mx + a_n b_m \cos nx \sin mx + b_n a_m \sin nx \cos mx + b_n b_m \sin nx \sin mx) \end{aligned}$$

(Note: in the above, n and m can be any letters, but we have to use two **different** letters since we're summing over both of them independently).

Now as before we integrate the above from $x = -\pi$ to π, and again we swap the sum and integral signs: the first term is a constant giving integral $(1/4)a_0^2 2\pi$; the next two terms contain only single sin's and cos's which all integrate to zero. Then in the double sum, we look up results from Eq.6.2 again, and all the terms with $m \neq n$ integrate to zero: so we can turn the double summation into a single summation with $m = n$ (think of summing over an infinite chessboard where all off-diagonal squares contain zeros). Then, the $\sin mx \cos nx$ terms also integrate to zero: finally the $\cos mx \cos nx$ terms and $\sin mx \sin nx$ terms (with $m = n$) integrate to π, so the overall result is

$$\int_{-\pi}^{\pi} f(x)^2 \, dx = \frac{1}{4} a_0^2 (2\pi) + 0 + 0 + \sum_{n=1}^{\infty} (a_n^2 \pi + 0 + 0 + b_n^2 \pi) \quad ;$$

this is Parseval's theorem as above.

In a very similar way, one can show that for two functions $f(x)$ and $g(x)$, with $f(x)$ having Fourier coefficients a_n, b_n and $g(x)$ having coefficients A_n, B_n, we obtain

$$\int_{-\pi}^{\pi} f(x) g(x) \, \mathrm{d}x = \tfrac{1}{2} \pi a_0 A_0 + \pi \sum_{n=1}^{\infty} (a_n A_n + b_n B_n) \; .$$

Example 6.5. Go back to the Fourier series for the square wave, Eq. 6.5 above. Putting this into both sides of Parseval's theorem, we have

$$\begin{aligned} \int_0^{\pi} 1 \, \mathrm{d}x &= \frac{\pi}{2} + \frac{4\pi}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \\ \pi &= \frac{\pi}{2} + \frac{4}{\pi} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) \end{aligned}$$

On rearranging we get

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

which we had already derived in another way in Example 6.3.

Parseval's theorem is important in practical applications, for example telling us numerically "how good" is an approximation to $f(x)$ given by taking only a finite number of terms in the Fourier series (as we have to do in real-world evaluation on a computer). We proceed as follows: define $S_N(x)$ to be the sum up to and including $n = N$ of the Fourier series for $f(x)$, then $S(x)$ is the infinite sum (the limit of $S_N(x)$ as N tends to infinity). If we define $E_N(x) = f(x) - S_N(x)$, this is the "residual error" if we keep only the first N terms of the series.

It is easy to see that the Fourier series for $E_N(x)$ has coefficients zero for $1 \le n \le N$, and a_n, b_n for $n > N$, so applying Parseval's theorem to $E_N(x)$,

$$\int_{-\pi}^{\pi} (E_N(x))^2 \, dx = \pi \sum_{n=N+1}^{\infty} (a_n^2 + b_n^2)$$

If we divide the above equation by the range 2π, the left-hand side becomes the mean value of E_N^2 over the range, which is the "mean square error" in our approximation $S_N(x)$. So, if the right-hand side is small, i.e. the sum of $a_n^2 + b_n^2$ is converging rapidly to its limit, we know that $S_N(x)$ is a good approximation of our original function $f(x)$.

6.3 Odd and even functions; Half range Fourier series

We recall the definitions of an "even" and "odd" function:
$f(x)$ is even $\Leftrightarrow f(x) = f(-x)$ for all x.
$f(x)$ is odd $\Leftrightarrow f(x) = -f(-x)$ for all x.

Any function $f(x)$ can always be written as

$$f(x) = \tfrac{1}{2}[f(x) + f(-x)] + \tfrac{1}{2}[f(x) - f(-x)] \, ,$$

in which the first bracket on the right is an even function and the second bracket is an odd function, by construction.

Since $\sin kx$ is odd and $\cos kx$ is even, we might suspect that for even functions $f(x)$ only cosine terms appear in the Fourier series (all $b_n = 0$), while similarly for odd functions only sine terms appear and all $a_n = 0$. This is correct, and we can easily check this, e.g.

$$\begin{aligned}
\pi a_n &= \int_{-\pi}^{\pi} f(x) \cos nx \, \mathrm{d}x \\
&= \int_{-\pi}^{0} f(x) \cos nx \, \mathrm{d}x + \int_{0}^{\pi} f(x) \cos nx \, \mathrm{d}x \\
&= \int_{u=\pi}^{0} f(-u) \cos(-nu) \, (-1) \mathrm{d}u + \int_{0}^{\pi} f(x) \cos nx \, \mathrm{d}x
\end{aligned}$$

where we have substituted $u = -x$ in the first half, so its range becomes π to 0. Now this is

$$\begin{aligned}
&= -\int_{0}^{\pi} f(-u) \cos nu \, (-1) \mathrm{d}u + \int_{0}^{\pi} f(x) \cos nx \, \mathrm{d}x \\
&= \int_{0}^{\pi} (f(-x) + f(x)) \cos nx \, \mathrm{d}x
\end{aligned}$$

where we have replaced u by $+x$ since it's a dummy variable. The above is clearly 0 if $f(x)$ is an odd function.

Similarly

$$\pi b_n = \int_{0}^{\pi} (f(x) - f(-x)) \sin nx \, \mathrm{d}x.$$

To summarise the above, if $f(x)$ is an even function, we have

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, \mathrm{d}x \, , \quad b_n = 0 \text{ for all } n \quad .$$

(where by symmetry we can halve the range of integration from 0 to π, and multiply by 2). And if $f(x)$ is an odd function , all $a_n = 0$, and

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, \mathrm{d}x \, .$$

We can use this property to make a Fourier series for a **half range** using **only** sine or only cosine terms, as follows. Suppose we are given a function $\phi(x)$ defined on $[0, \pi]$ (a "half range"), then we can define two new functions on the range $[-\pi, \pi]$: we construct an even function $f(x)$ such that $f(x) = \phi(x)$ in $(0, \pi)$ and $f(x) = \phi(-x)$ if $-\pi < x < 0$. Likewise we define an odd function $g(x)$ such that $g(x) = \phi(x)$ for $0 \le x < \pi$, and $g(x) = -\phi(-x)$ if $-\pi < x < 0$.

Note that **both** $f(x)$ and $g(x)$ are equal to $\phi(x)$ on the range $(0, \pi)$, but they have opposite signs on the range $(-\pi, 0)$. (Note also that $h(x) = \frac{1}{2}(f(x) + g(x))$ is equal to $\phi(x)$ on $(0, \pi)$ and zero on $(-\pi, 0)$).

Inserting these $f(x)$ and $g(x)$ into Eq. 6.1, our even function $f(x)$ gives a Fourier series with

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} \phi(x) \cos nx \, \mathrm{d}x \, , \qquad b_n = 0,$$

and the odd function $g(x)$ gives a Fourier series with

$$a_n = 0\ , \qquad b_n = \frac{2}{\pi}\int_0^{\pi} \phi(x)\sin nx\, \mathrm{d}x\ .$$

These are called respectively the **half-range cosine series** and **half-range sine series** for $\phi(x)$; both of those series are equal to $\phi(x)$ on the range $(0,\pi)$, but they have opposite signs on the range $(-\pi,0)$.

(Also it is clear that if you take the average of the above two series, you get the series for $h(x)$ above, which is equal to $\phi(x)$ on $(0,\pi)$ and zero on $(-\pi,0)$).

Example 6.6. $f(x)$ is such that $f(x) = f(x+2\pi)$ and $f(x) = -f(-x)$, and on $0 \le x \le \pi$, $f(x) = x(\pi - x)$. Find its Fourier series, and prove that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} + \ldots = \frac{\pi^3}{32}$$

The given $f(x)$ has period 2π and is odd, so we know the series contains only sine terms, and

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left\{\left[-x(\pi-x)\frac{\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi}(\pi-2x)\frac{\cos nx}{n}\,\mathrm{d}x\right\} \\
&= \frac{2}{\pi}\left\{\left[(\pi-2x)\frac{\sin nx}{n^2}\right]_0^{\pi} + 2\int_0^{\pi}\frac{\sin nx}{n^2}\,\mathrm{d}x\right\} \\
&= \frac{4}{\pi}\left[-\frac{\cos nx}{n^3}\right]_0^{\pi} \\
&= \begin{cases} 0 & \text{for } n = 2k, \\ \dfrac{8}{\pi(2k+1)^3} & \text{for } n = 2k+1. \end{cases}
\end{aligned}$$

Thus

$$f(x) = \frac{8}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{(2k+1)^3}. \tag{6.7}$$

To get the series requested, we try evaluating (6.7) at some x such that $\sin(2k+1)x = (-1)^k$. This occurs at $x = \pi/2$. Evaluating both sides there gives

$$f(\pi/2) = \frac{\pi^2}{4} = \frac{8}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}$$

which on rearranging gives the required result.

6.4 Arbitrary range Fourier series

Here we extend the Fourier series to the case when the range of our function is not $-\pi \le x \le \pi$. If we have $f(x)$ defined in a range $-L \le x \le L$, instead of $-\pi < x \le \pi$, then we can define a new variable $y \equiv \pi x/L$ (a rescaled version of x), so that $-\pi \le y \le \pi$ and write f as a Fourier series in y.

$$\begin{aligned}
f(x) &= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos ny + b_n \sin ny) \\
&= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L})).
\end{aligned}$$

where

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{y=-\pi}^{y=\pi} f(\frac{Ly}{\pi})\cos\frac{n\pi x}{L}\mathrm{d}(\frac{\pi x}{L}),\\
&= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\mathrm{d}x.
\end{aligned}
$$

and similarly

$$
b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\mathrm{d}x.
$$

Here we have just "rescaled": observe that as x goes from $-L$ to L, the quantity $n\pi x/L$ goes from -πn to $+\pi n$ so there are again an integer n "wiggles" in each cos/sin term.

For functions which are a simple stretch/squash of another function whose Fourier series we have already worked out, we can rescale variables.

Example 6.7. Find the Fourier series for the function $g(x)$ of period $2c$ such that

$$
g(x) = \begin{cases} 0 & \text{if } -c < x < 0 \\ x & \text{if } 0 < x < c. \end{cases}
$$

Using the result of example 6.1, replacing x by y, we have

$$
\begin{aligned}
f(y) &= \frac{\pi}{4} - \sum_{k=1}^{\infty}\frac{2}{\pi(2k+1)^2}\cos(2k+1)y + \sum_{1}^{\infty}\frac{(-1)^{n+1}}{n}\sin ny \qquad -\pi < y \le \pi\\
\Rightarrow f(\frac{\pi x}{c}) &= \frac{\pi}{4} - \sum_{k=1}^{\infty}\frac{2}{\pi(2k+1)^2}\cos\frac{(2k+1)\pi x}{c} + \sum_{1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{c} \qquad -c \le x \le c
\end{aligned}
$$

But we have $f(\pi x/c) = 0$ for $-c < x < 0$, or $\pi x/c$ for $0 < x \le c$, so $f(\pi x/c) = (\pi/c)g(x)$ for all $-c < x \le c$. So we just multiply the series above by c/π, and get

$$
\Rightarrow g(x) = \frac{c}{4} - \sum_{k=1}^{\infty}\frac{2c}{\pi^2(2k+1)^2}\cos\frac{(2k+1)\pi x}{c} + \sum_{1}^{\infty}\frac{(-1)^{n+1}c}{\pi n}\sin\frac{n\pi x}{c}
$$

Appendix

This section will not be lectured and is not for examination

The following shows the kind of application Fourier himself had in mind and gives an example of some methods in partial differential equations which we will meet in another context in the next chapter.

Example 6.8. In the propagation of heat in a solid in one dimension, the temperature θ obeys the equation

$$
k\frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial t}.
$$

This is the simplest case of the **diffusion equation**.

We introduce here a new idea which will run through the rest of the course. This is **separation of variables**: we can see that if we look for a solution in the form $X(x)T(t)$ we will find

$$
kT\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = X\frac{\mathrm{d}T}{\mathrm{d}t} \Rightarrow \frac{k}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = \frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t}.
$$

Here the left side depends only on x and the right side only on t: hence the two sides must both equal the same constant (only a constant can depend only on x, and only on t, at the same time). We then have two equations

$$k\frac{\mathrm{d}^2X}{X\,\mathrm{d}x^2} = \lambda, \qquad \lambda = \frac{\mathrm{d}T}{T\,\mathrm{d}t},$$

to solve, where λ is our unknown constant. When we have solved these, we multiply the answers together to solve the original equation. In general we assume (and indeed usually we can prove) that the full solution is a (possibly infinite) sum of solutions of the separable type.

For Fourier's problem we proceed as follows:
At the earth's surface, the temperature θ is assumed to vary periodically over the year (for simplicity) so it has a Fourier series in time t with period 1 year. We define x to be the depth into the earth. Then at the surface $x = 0$ we can write

$$\theta = \tfrac{1}{2}a_0 + \sum_{n=1}(a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T})$$

with $T = 365/2$ days.

Now at other x we let a_n and b_n depend on x and put these into the differential equation: this means we are writing the whole solution as a sum of separable solutions in which the t dependence gives a Fourier series (with different coefficients at each x). Plugging this into the original equation and equating coefficients in the Fourier series we get

$$k\frac{\partial^2 a_n}{\partial x^2}\cos\frac{2n\pi t}{T} = \frac{2n\pi}{T} b_n \cos\frac{2n\pi t}{T}.$$

$$k\frac{\partial^2 b_n}{\partial x^2}\sin\frac{2n\pi t}{T} = -\frac{2n\pi}{T} a_n \sin\frac{2n\pi t}{T}.$$

These can be written as a single complex equation

$$\frac{\partial^2(b_n + \mathrm{i}a_n)}{\partial x^2} = \frac{2n\pi\mathrm{i}}{T}(b_n + \mathrm{i}a_n).$$

This equation is easy to solve as it is a linear equation with constant coefficients. [For those who have done the Differential Equations course, the auxiliary equation has roots

$$\pm\sqrt{\frac{n\pi}{kT}}(1+i)$$

and that gives the solutions. We need the solution with a negative real part (temperature variation decreases as we go into the earth).] The solution is

$$b_n + \mathrm{i}a_n = c\exp\left(-\sqrt{\frac{n\pi}{kT}}(1+i)x\right),$$

for some constant c. This means we have a solution which varies sinusoidally with time, but the amplitude of variation decreases by a factor e in a distance $\sqrt{kT/n\pi}$. Some realistic figures are $k = 2.10^{-3}$ cm^2/s, $T = 365.24.3600/2$ secs, giving $1/\lambda \equiv \sqrt{kT/\pi} = 177$ cm for annual variation and roughly $1/19$ of this for daily variation. The amplitude of the annual variation halves in a distance x such that $\lambda x = \ln 2$, about 123 cm. So in 5 metres the variation of temperature reduces by a factor 1/16 (it also turns out that at that depth the variation is out of phase with the surface, i.e. coolest in mid-summer).

6.5 Fourier Transforms

This section is not examinable, but is included since it may be useful for later courses.

To conclude this chapter, it is worth a quick look at the extension of Fourier series to Fourier Transforms. The principle remains the same, i.e. expressing a general function as a sum of trigonometric functions of different frequency.

There are two main steps to get from Fourier series to Fourier transforms: firstly, we introduce complex numbers and use Euler's formula

$$e^{inx} = \cos nx + i \sin nx \qquad .$$

Then we change the definition of the Fourier series to

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

and the coefficients c_n become

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

What we have done here is just make the coefficients c_n complex, extended the infinite sum to negative integers n, and changed the prefactor from $1/\pi$ to $1/(2\pi)$ to compensate for doubling the number of terms in the sum. (The $n = 0$ case does not have positive and negative terms so the half in Eq. 6.1 gets absorbed in the above). In this case we can easily see, taking real and imaginary parts of the above, that $c_n = \frac{1}{2}(a_n - ib_n)$ where the a_n, b_n are the same as previous sections; assuming $f(x)$ is real-valued, then it is clear from the definition that $c_{-n} = \frac{1}{2}(a_n + ib_n) = \bar{c_n}$, the complex conjugate.

This has not really done anything very new, it just turns two real formulae for a_n, b_n into one complex formula for c_n. The real parts of the c_n's are the cosine terms and the imaginary parts give the sine terms; if we extract the two terms for $+n$ and $-n$ in the series for $f(x)$ we have

$$\begin{aligned} c_n e^{inx} + c_{-n} e^{-inx} &= \tfrac{1}{2}(a_n - ib_n)(\cos nx + i \sin nx) + \tfrac{1}{2}(a_n + ib_n)(\cos nx - i \sin nx) && (6.8) \\ &= (a_n \cos nx + b_n \sin nx) \quad , && (6.9) \end{aligned}$$

so the imaginary parts cancel, and this agrees with what we had before.

This also allows us to extend the formula to complex-valued $f(x)$, in which case the terms $c_n + \bar{c}_{-n}$ are no longer real, and their imaginary parts give the complex part of $f(x)$.

To extend to Fourier transforms, we generalise the above to the arbitrary-range series, i.e. let $f(x)$ be periodic with period L, i.e.

$$\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} c_n e^{-2\pi i n x/L} \\ c_n &= \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{2\pi i n x/L}\, dx \end{aligned}$$

Now if we write $\delta = 2\pi/L$ and $\omega_n = n\delta$ this becomes

$$\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} c_n e^{i\omega x} \\ c_n &= \frac{\delta}{2\pi} \int_{-L/2}^{L/2} f(x) e^{i\omega x}\, dx \end{aligned}$$

and if we let the range L tend to infinity, let $d_n = c_n/\delta$, let δ tend to zero, we can convert the infinite discrete series of coefficients d_n into a continuous function $\mathscr{F}(\omega)$, and (skipping some details) we arrive at

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \mathscr{F}(\omega) e^{i\omega x}\, d\omega \\ \mathscr{F}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \qquad . \end{aligned} \tag{6.10}$$

Here $\mathscr{F}(\omega)$ is called the **Fourier transform** of $f(x)$, with ω called the (angular) frequency , which is the continuous version of the n we had before.

Note: there are several possible "arbitrary choices" of where to put the 2π's and minus signs in the above definitions; some books put a factor $1/\sqrt{2\pi}$ before both integrals, which makes them symmetrical. Other authors leave a 2π inside the exponential term, in which case ω is usually changed to a different letter e.g. $\nu = \omega/2\pi$. As long as this is done consistently, it doesn't matter, but there must be factors of 2π somewhere in the definitions.