Queen Mary
UNIVERSITY OF LONDON

B.Sc. EXAMINATION BY COURSE UNITS

Answers to 2007 MAS204 Calculus III exam

General comment: this examination paper was too hard, as it turned out. Allowance was made for this in marking. On the other hand, some section A questions induced surprisingly many basic errors. For those taking the course in 2007-8, note that questions A.6 and B.5 are no longer on the syllabus.

SECTION A

Answers in section A are cross-referenced to the Key Objectives (KO), in the order used in the course (Copy appended to these answers).

A1. {KO4 and KO7: similar problems in lectures and exercises}

(a) $\nabla V = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$ [3]
Comment: a disappointingly high fraction of candidates gave a scalar rather than a vector as the answer. (Those made this mistake and differentiated correctly gave $(2x + 2y - 1)$ *but still got no marks.)*

(b) A paraboloid of revolution about the z-axis [3]
Comment: most of the wrong answers said it was a sphere.

(c) At $\mathbf{P}$, $\nabla V = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, so we get $\mathbf{r}.\nabla V = \mathbf{P}.\nabla V$ which is
$2x + 4y - z = 2.1 + 4.2 - 4 = 6.$ [4]
Comment: some candidates forgot that before using the gradient in the equation for the plane it had to be evaluated at $\mathbf{P}$*. Hence they gave a quadratic, rather than linear, equation. Some claimed* $\nabla V = 2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$.

A2. {KO 3: similar to examples in lectures and courseworks.}

(a) $\mathrm{d}\mathbf{r} = (-2t\mathbf{i} + \mathbf{j})\mathrm{d}t$ so we need

$$\int_{-2}^{2}(-2t(4-t^2) + 2t^2 + 2t(4-t^2))\mathrm{d}t = \int_{-2}^{2} 2t^2\mathrm{d}t = [2t^3/3]_{-2}^{2} = 32/3.$$

[4]

Comment: quite well done. Main errors were in collecting the terms to get $2t^2$.

(b) Taking $y = 2\cos\theta$, $z = 2\sin\theta$ [or equivalent]
$\mathrm{d}\mathbf{r} = (-2\sin\theta\mathbf{j} + 2\cos\theta\mathbf{k})\mathrm{d}\theta$, so we have

$$\int_0^{\pi} 4\cos^2\theta\mathrm{d}\theta = 4\tfrac{1}{2}(\pi) = 2\pi.$$

[This question continues overleaf ...]

[4]

Comment: given the number of times students had seen this kind of thing in lectures and exercises, surprisingly few knew how to start by choosing a good parametrization.

A3. {KO5 and 7. Bookwork and simple application.}
The two surfaces have the same boundary described in the same sense. Hence by Stokes' theorem, the two surface integrals of the curls are the same. [3]

The given $\mathbf{F}$ has curl $\mathbf{i}+\mathbf{j}+\mathbf{k}$. (Candidates could evaluate this using the determinant form.) [3]

Hence, using Stokes' theorem to convert to the integral over the disk, the integral over the surface is $\int \mathrm{d}x\mathrm{d}y$ over the disc which is πa^2. [The integral directly round the curve is tractable so candidates are specifically told not to do that.] [3]
Comment: A number of candidates worked on $\nabla(x^2+y^2+z^2-a^2)$. Rather few gave the (short) answer to the first part.

A4. {KO6. Seen: this example done in coursework.}

$$\begin{aligned}
\epsilon_{ijk}\partial_j(\epsilon_{klm}F_l G_m) &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\partial_j(F_l G_m) \\
&= \partial_m(F_i G_m) - \partial_l(F_l G_i) \\
&= F_i(\partial_m G_m) + G_m\partial_m F_i - G_i(\partial_m F_m) - F_m\partial_m G_i
\end{aligned}$$

[Forming expression 2, using identity and delta 3]
so $\nabla\times(\mathbf{F}\times\mathbf{G}) = \mathbf{F}(\nabla\cdot\mathbf{G}) + (\mathbf{G}.\nabla)\mathbf{F} - \mathbf{G}(\nabla\cdot\mathbf{F}) - (\mathbf{F}.\nabla)\mathbf{G}$ [last steps 2]
Comment: a lot of students started by writing down $\epsilon_{ijk}\partial_j(\epsilon_{ilm}F_l G_m)$. I guess they thought the names of indices in the problem had to exactly match those in the hint, despite having done the coursework.

A5. {KO 1: similar examples in coursework and old papers}
f is odd so we need only

$$b_n = \frac{2}{\pi}\int_0^\pi \sin(nx)\mathrm{d}x = \frac{2}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^\pi = \frac{2}{n\pi}[1-(-1)^n]$$

which is zero if n is even and $4/n\pi$ if n is odd.

Hence the series is as given. [6]
Comment: quite a few candidates wasted time by directly evaluating a_n rather than using the oddness of f.

Since $f^2 = 1$, Parseval gives

$$2\pi = 16\pi \sum_{n\ \mathrm{odd}} \frac{1}{n^2\pi^2}$$

which is easily rearranged to the given formula. [3]
Comment: those who tried this did it well.

[Next question overleaf]

A6. {KO 2: First part bookwork. Rest unseen}
The condition is that L is independent of y. [2]
Comments: quite a few said it had to be independent of y'.

In this problem that is true so we have $2x^{2n}y'$ =constant, whence $y = Ax^{1-2n} + B$. [4]
Comment: done well

For this to cross $x = 0$ we need $A = 0$ or $n < 1/2$. [2]
[Bonus 2 for anyone who did $n = 1/2$ separately!]
Comment: answered by very few

A7. {KO8. First part unseen but like bookwork. Second like one in lectures.}

$$0 = \frac{\partial^2\Phi}{\partial^2 x} + \frac{\partial^2\Phi}{\partial^2 y} = \sinh(2\pi y)\frac{\partial^2 g}{\partial^2 x} + 4\pi^2\sinh(2\pi y)g \Rightarrow \frac{\partial^2 g}{\partial^2 x} = -4\pi^2 g.$$

The solution is $g = A\cos(2\pi x) + B\sin(2\pi x)$. [5]

The boundary condition at $y = 0$ is OK, the ones at $x = 0$ and $x = 1$ are OK if $A = 0$ and then $B = 1/\sinh(4\pi)$ to get the one on $y = 2$. [4]
Comment: some answers assumed the first part was **exactly** *like the bookwork, i.e. did not read the question carefully. Very few could do the last part.*

SECTION B

B1. {Unseen but closely related to examples on past papers}

(a) $(\nabla \times \mathbf{F}) \times \mathbf{F} = (f\mathbf{F}) \times \mathbf{F} = \mathbf{0}$ by usual rule that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector. [3]

(b) The initial equation implies (since div(curl) is always 0, and using vector identities) that
$\nabla \cdot (\nabla \times \mathbf{F}) = 0 = \nabla \cdot (f\mathbf{F}) = \mathbf{F}.\nabla f + f\nabla \cdot \mathbf{F}$ so if $\nabla \cdot \mathbf{F} = 0$ then $\mathbf{F}.\nabla f = 0$. [5]

(c) If f is constant, $\nabla \times (\nabla \times \mathbf{F}) = f(\nabla \times \mathbf{F})$ – also Beltrami (by applying curl to $\nabla \times \mathbf{F} = f\mathbf{F}$). [3]

The given $\mathbf{F}$ has a curl given by

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A\sin(y^3) & 0 & A\cos(y^3) \end{vmatrix} \\ &= 3Ay^2(-\sin(y^3)\mathbf{i} - \cos(y^3)\mathbf{k}) \end{aligned}$$

so $f = -3y^2$. [4+2]
Showing $\nabla \cdot \mathbf{F} = 0$ may be done directly but the simplest way is to use the argument in part (b) in reverse since it is easy to see $\mathbf{F}.\nabla f = 0$ in this case and so $f\nabla \cdot \mathbf{F} = 0$ but $f \neq 0$. [3]
Comment: quite well done, with slips in calculation the main errors. Some assumed, wrongly, that $(\nabla \times \mathbf{F}) \times \mathbf{F} = \nabla \times (\mathbf{F} \times \mathbf{F})$.

[Next question overleaf]

B2. In spherical polars

$$\begin{aligned}
\nabla\cdot\mathbf{F} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial(r^2\sin\theta F_r)}{\partial r}+\frac{\partial(r\sin\theta F_\theta)}{\partial\theta}+\frac{\partial(rF_\phi)}{\partial\phi}\right]\\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial(r^3\sin\theta(\sin\theta+\cos\theta))}{\partial r}+\frac{\partial(2r^2\sin^2\theta)}{\partial\theta}\right]\\
&= \frac{1}{r^2\sin\theta}[3r^2(\sin^2\theta+\sin\theta\cos\theta)+4r^2\sin\theta\cos\theta]\\
&= 3\sin\theta+7\cos\theta.
\end{aligned}$$

(using the formulae on the cover sheet) [4]

The volume integral is

$$\begin{aligned}
\int(\nabla\cdot\mathbf{F})\mathrm{d}V &= \int_0^a\int_0^\pi\int_0^{2\pi}(3\sin\theta+7\cos\theta)r^2\sin\theta\,\mathrm{d}\phi\mathrm{d}\theta\mathrm{d}r\\
&= \int_0^a\int_0^\pi\int_0^{2\pi}(3\sin^2\theta+\frac{7}{2}\sin(2\theta))r^2\,\mathrm{d}\phi\mathrm{d}\theta\mathrm{d}r\\
&= (2\pi)\left(\frac{a^3}{3}\right)\left(\tfrac{3}{2}\pi+\tfrac{7}{4}[\cos(2\theta)]_0^\pi\right)\\
&= \pi^2a^3.
\end{aligned}$$

The surface integral is (since $\mathrm{d}\mathbf{S}=r^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi\,\mathbf{e}_r$)

$$\begin{aligned}
\int\mathbf{F}.\mathrm{d}\mathbf{S} &= \int_0^\pi\int_0^{2\pi}a^3(\sin^2\theta+\cos\theta\sin\theta)\mathrm{d}\theta\mathrm{d}\phi\\
&= \int_0^\pi\int_0^{2\pi}a^3(\sin^2\theta+\tfrac{1}{2}\sin(2\theta))\mathrm{d}\theta\mathrm{d}\phi\\
&= 2\pi a^3(\tfrac{1}{2}\pi+\tfrac{1}{4}[\cos(2\theta)]_0^\pi)\\
&= \pi^2a^3
\end{aligned}$$

{2+3 for forming integrals, 3 for correct integration of $\sin^2$, 3 for $\sin\cos$ integration, 2 for final answers} [13]

The hemisphere gives simply half the volume integral for the sphere but one could also do it by halving the surface integral since $\mathbf{F}.\mathrm{d}\mathbf{S}=0$ on the plane face. [3]

Comment: a lot of correct evaluations of $\nabla\cdot\mathbf{F}$ but very few good answers for the rest.

B3. {Unseen. First two parts like bookwork and examples. Last bit novel but rescaling x in a Fourier series seen in an example.}

(a) Candidates could calculate the curl and show it is zero, or could integrate directly to get
$\Phi=\sum_{n=1}^{\infty}\frac{aA_n}{n\pi}\sin(n\pi x/a)\sinh(n\pi y/a)$. [8]

[This question continues overleaf ...]

(b)

$$\begin{aligned}\nabla\cdot\mathbf{F} &= \frac{\partial G}{\partial x}+\frac{\partial H}{\partial y}\\ &= \sum_{n=1}^{\infty}-\frac{n\pi A_n}{a}\sin(n\pi x/a)\sinh(n\pi y/a)\\ &\quad+\sum_{n=1}^{\infty}\frac{n\pi A_n}{a}\sin(n\pi x/a)\sinh(n\pi y/a)\\ &= 0.\end{aligned}$$

[4]

(c) The given form already satisfies the boundary conditions on $x = 0$, $x = a$, and $y = 0$. [3]
The remaining boundary condition can be found by rescaling the x (by $X = \pi x/a$) in the given formula to get

$$\tfrac{1}{2}a - |\tfrac{1}{2}a - x| = \frac{4a}{\pi^2}\sum_{p=1}^{\infty}\frac{(-1)^p\sin((2p+1)\pi x/a)}{(2p+1)^2}.]$$

whence

$$\sum_{n=1}^{\infty}A_n\sin(n\pi x/a)\cosh(n\pi b/a) = \frac{4a}{\pi^2}\sum_{p=1}^{\infty}\frac{(-1)^p\sin((2p+1)\pi x/a)}{(2p+1)^2}.]$$

so we need $A_n = 0$ for n even and [5]

$$A_{2p+1} = \frac{4a(-1)^p}{\pi^2(2p+1)^3\cosh((2p+1)\pi b/a)}.$$

Comment: very few answers and those mostly not very complete.

B4. {Unseen but using basic Fourier properties}
The coefficients in the Fourier series are given by

$$\begin{aligned}a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\mathrm{d}x\\ &= \frac{1}{\pi}\int_0^{\pi}\cos x\cos nx\mathrm{d}x\\ &= \frac{1}{2\pi}\int_0^{\pi}(\cos(n+1)x+\cos(n-1)x)\mathrm{d}x\\ &= \frac{1}{2\pi}\left[\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right]_0^{\pi}\\ &= 0\end{aligned}$$

[This question continues overleaf . . .]

if $n \neq 1$, and $\frac{1}{\pi}\int_0^\pi \cos^2 x \mathrm{d}x = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}$ if $n = 1$. [6]

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} \cos x \sin nx \mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^{\pi} (\sin(n+1)x + \sin(n-1)x)\mathrm{d}x \\ &= \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \\ &= \frac{n}{\pi(n^2-1)}(1-(-1)^{n+1}) \end{aligned}$$

if $n \neq 1$. This is 0 if n is odd and $\frac{4p}{\pi(4p^2-1)}$ if $n = 2p$ is even. If $n = 1$ we have

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \cos x \sin x \mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} \tfrac{1}{2}\sin(2x)\mathrm{d}x = \frac{1}{\pi}\tfrac{1}{4}[-\cos(2x)]_0^{\pi} = 0.$$

[6]

[Note: a really smart candidate might say that $f - \frac{1}{2}\cos x$ is odd so only has a sine series, etc.]

The value of S at $x = \pi$ is $-\frac{1}{2}$. (Jump from -1 to 0 there.) [2]

Evaluation at $x = \pi/4$ gives [method 3]

$$\frac{1}{\sqrt{2}} = \tfrac{1}{2}\frac{1}{\sqrt{2}} + \frac{4}{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s(2s+1)}{(4(2s+1)^2-1)},$$

since $\sin(2px)$ at $x = \pi/4$ is $\sin(\frac{1}{2}p\pi)$ which is 0 for even p and $(-1)^{(p-1)/2}$ for odd p. Rearranging gives the result. [3]

Comment: a surprising number of candidates evaluated using $f = x$ rather than $f = \cos x$. Some wrote $a_0 = \cos x$. Few were clear about a_1 being a special case. Some evaluated S at $\pi/4$ directly, and correctly. Few got all the way.

B5. {First part bookwork. Example unseen but similar to examples seen.}
y must obey the Euler-Lagrange equation, i.e. [3]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0 \ .$$

$$\text{Since} \quad \frac{\mathrm{d}L}{\mathrm{d}x} = \frac{\partial L}{\partial x} + y'\frac{\partial L}{\partial y} + y''\frac{\partial L}{\partial y'} = y'\frac{\partial L}{\partial y} + y''\frac{\partial L}{\partial y'} \ , \qquad [3]$$

[This question continues overleaf . . .]

if we multiply the Euler-Lagrange equation by y' we get

$$\begin{aligned} 0 &= y'\frac{\mathrm{d}}{\mathrm{d}x}(\frac{\partial L}{\partial y'}) - y'\frac{\partial L}{\partial y} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(y'\frac{\partial L}{\partial y'}) - y''\frac{\partial L}{\partial y'} - y'\frac{\partial L}{\partial y} \qquad [2] \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(y'\frac{\partial L}{\partial y'} - L)\ . \qquad [2] \end{aligned}$$

Hence in this case

$$y'\frac{\partial L}{\partial y'} - L = \text{constant}\ .$$

The Hamiltonian first integral is $(y')^2 + k^2y^2 = \text{constant}$ [4]
which gives $y = A\sin(kx+B)$ (or some equivalent) and the conditions give $B = 0$, [4]
$A = 5$ whence $y = 5\sin(kx)$. [2]
This part can also be done using the Euler-Lagrange equation itself.
Comment: this one is deliberately slightly easy because students found this a hard topic. Those who tried it did quite well, though the proofs were sometimes not well expressed.

B6. {First part is rearranged bookwork. Second part is unseen but on similar lines to problems in lectures and coursework.}

(a) Solving the S equation with $\lambda = -m^2$ gives $S = A\sinh(m\phi) + B\cosh(m\phi)$, and no such function has $S(0) = S(2\pi)$ (except the trivial one $S = 0$). Hence $\lambda \geq 0$. [3]

(b) For $\lambda = m^2 > 0$, $S = A\cos(m\phi) + B\sin(m\phi)$ and $R = C\rho^m + D\rho^{-m}$.
For $\lambda = 0$, $S = A\phi + B$ and $R = C\ln\rho + D$. [6]

In the problem, writing $4\cos^2\phi = 2(\cos(2\phi) + 1)$ we see that only terms with $m = 0$, $m = 2$ and $m = 3$ occur in the boundary conditions. So we can guess this is all we need in the answer. [3]
Boundedness at the origin implies we do not need the ρ^{-m} or $\ln\rho$ terms. [2]
Single-valuedness eliminates the $A\phi$ terms, [1]
so we have $A_0 + \rho^2(A_2\cos(2\phi) + B_2\sin(2\phi)) + \rho^3(A_3\cos(3\phi) + B_3\sin(3\phi))$
and matching with the given values $2+2\cos(2\phi)+\sin(3\phi)$ at $\rho = 2$ gives us $B_2 = A_3 = 0$, $A_0 = 2$, $A_2 = \frac{1}{2}$, $B_3 = \frac{1}{8}$, so

$$\Phi = 2 + \tfrac{1}{2}\rho^2\cos(2\phi) + \tfrac{1}{8}\rho^3\sin(3\phi).$$

[5]

Comment: almost nobody tried this, probably because sinilar questions had not appeared for a year or two. I hope students in later years will do more practice on this part of the course.

[Next question overleaf]

KEY OBJECTIVES of the course

The student should

1. Know the important properties of Fourier series and be able to compute coefficients.
2. Be able to write down, in simple cases, variational integrals for curves $y(x)$ and derive and solve their Euler-Lagrange equations.
3. Be able to do simple line and surface integrals. (E.g. Evaluate $\int \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ for a given vector field, with the path given in either parametric or non-parametric form.)
4. Be able to do simple manipulations involving gradient, divergence, and curl, and understand their geometrical/physical meaning.
5. Understand Stokes' theorem and the divergence theorem and be able to do simple problems applying these.
6. Be able to do simple manipulations in index notation, and switch between vector and index notation wherever necessary.
7. Understand three-dimensional cartesian, cylindrical, and spherical polar coordinates geometrically, and be able to express lines, surfaces, and volumes in coordinate or vector notation as appropriate.
8. Understand the variable-separation technique for PDEs and be able to do simple solution problems with Laplace's equation in (at least) 2D Cartesian coordinates.

[End of examination paper]