S357/B

The Open University

Course Examination 2005

Space, Time and Cosmology

Monday 10 October 2005 2.30pm – 5.30pm

Time allowed: 3 hours

This paper is divided into **TWO** parts, I and II. Part I carries 48 per cent of the total marks, and Part II carries 52 per cent. You are advised to spend roughly equal time on each part. Remember to allow time for reading the question paper first, and your answers afterwards. You will not be given extra time to do this.

You should answer **ALL** the questions in Part I, and **FOUR** questions from Part II.

Record your answers to Part I in the spaces provided in the green question and answer booklet inserted in this question paper. Answer **each** question attempted from Part II in a **separate** answer book (i.e. use **four** answer books in all). *It is most important that you indicate in the box provided on the front of each answer book the number of the question which you have answered in that book.*

At the end of the examination

1. Make sure that you have written your personal identifier and examination number on Part I of the question paper and on **each** answer book used. **Failure to do so will mean that your work cannot be identified.**
2. Complete the grid on the front of the enclosed green booklet containing Part I.
3. Put all your used answer books and your question paper together with your signed desk record on top. Fix them all together with the paper fastener provided.

Note: Part I of this paper is provided as a separate insert.

PART II (*i*) *Part II carries 52 per cent of the total marks, and you are advised to spend about* **90 minutes** on it.

(*ii*) *Attempt* **FOUR** *questions only.*

(*iii*) *All answers carry thirteen marks, and where questions are divided into more than one section (labelled* (*a*), (*b*), (*c*), *etc.), the marks allocated to each section are indicated.*

(*iv*) *Write your answer to* **each** *question from Part II in a* **separate** *answer book. You should therefore submit* **FOUR** *answer books for this part.*

Question 13 This question concerns Newton's laws in a two-particle system and modifications to these laws made by Einstein.

At time t, particle 1, with mass m_1, has position $\mathbf{x}_1$, velocity $\mathbf{v}_1$, momentum $\mathbf{p}_1$, and acceleration $\mathbf{a}_1$. It is acted on by a force $\mathbf{F}_1$ exerted by particle 2.

Similarly, particle 2, with mass m_2, has position $\mathbf{x}_2$, velocity $\mathbf{v}_2$, momentum $\mathbf{p}_2$, and acceleration $\mathbf{a}_2$. It is acted on by a force $\mathbf{F}_2$ exerted by particle 1.

(*a*) Write down the relation between $\mathbf{v}_1$ and $\mathbf{p}_1$ and the relation between $\mathbf{a}_1$ and $\mathbf{F}_1$, within the framework of Newtonian mechanics. (*2 marks*)

(*b*) State how the first relation is modified in the special theory of relativity. Name a key concept in the general theory of relativity that enables one to dispense with the second relation when the interaction between the particles is purely gravitational. (*2 marks*)

(*c*) Write down a relation between the vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ that is asserted by that part of Newton's third law which ensures conservation of total linear momentum. Show that this relation leads to the constancy of $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$. (*3 marks*)

(*d*) Define the total angular momentum of the system and state the full conditions on the force vectors that lead to conservation of angular momentum. (*3 marks*)

(*e*) In the special theory of relativity, the total momentum of *material* particles is not conserved, in general. Yet the total momentum of the *entire* system is conserved. What else carries momentum in Einstein's account of electrodynamics? How was this taken into account in the field equations of general relativity? What was the ratio of the pressure and energy density when our Universe was a few minutes old? (*3 marks*)

Question 14 This question concerns the Doppler effect in special relativity. Commander C dispatches two identical spaceships, A and B, in diametrically opposite directions. Each recedes from C at speed $v = \frac{1}{2}c$. Take care in this question to avoid non-relativistic approximations. Assume that the distances of A and B from C are small enough for you to neglect effects of spacetime curvature.

(*a*) Spaceship A transmits a radio signal, at frequency f_0, as measured in the rest frame of A. At what frequency, f_1, will the signal be received by C? (*3 marks*)

(*b*) Commander C re-transmits the signal at the same frequency f_1 that it was received. At what frequency, f_2, will it be received by B? Give your answer in terms of f_1. (*3 marks*)

(*c*) Since the intervention of C has no effect on the frequency, B may use the ratio f_2/f_0 to determine the speed u at which A is receding from B. Show that $u = \frac{4}{5}c$, using only the relativistic Doppler formula. (*3 marks*)

(*d*) Suppose that C is at rest in the frame in which the cosmic microwave background radiation (CMBR) is isotropic and has temperature T_0. What are the maximum and minimum temperatures of the CMBR that A and B can detect? In which directions will these temperatures be detected? Which special-relativistic effect explains the fact that each of A and B detects a temperature *less* than T_0 in a direction at right angles to the line joining A to B? (*4 marks*)

Question 15 This question concerns a planet P with mass M. The events of interest occur on a radial line connecting the centre of P to a distant observer O, at fixed spatial coordinates. The Schwarzschild metric, exterior to P, gives

$$(c\,\Delta\tau)^2 = \left(1 - \frac{k}{r}\right)(c\,\Delta t)^2 - \frac{(\Delta r)^2}{1 - k/r}$$

where $\Delta\tau$ is the proper time interval between events whose (t, r) coordinates differ by the small amounts $(\Delta t, \Delta r)$ and $k = 2GM/c^2$. Clock C_1 sits on the surface of P, at radial coordinate $r = R \gg k$, on the line connecting the centre of P to O. It is identical to another clock C_0, held by O. In this question, you may use the approximations

$$\sqrt{1-x} \approx 1 - \frac{1}{2}x$$

and

$$\frac{1}{1+x} \approx 1 - x$$

when $x \ll 1$.

(*a*) Derive the approximation

$$\frac{f_1}{f_0} \approx 1 - \frac{GM}{Rc^2}$$

for the ratio of the frequencies f_1 and f_0 of clocks C_1 and C_0, according to O. (*4 marks*)

(*b*) Now consider another clock, C_2, identical to C_1 and C_0 and located on the line between them, at a small height $h \ll R$ above C_1. Let f_2 be the frequency of clock C_2 according to O. Show that

$$\frac{f_2 - f_1}{f_0} \approx \frac{gh}{c^2}$$

where $g = GM/R^2$. Cite an experiment performed on Earth which confirmed this effect, at 1% precision, and a basic principle of general relativity that allows one to predict its result using Newton's approximate theory of gravity. (*5 marks*)

(*c*) A light signal is sent from C_1 to C_2. Derive the rate at which its radial coordinate r increases with t and show that this is slightly less than c. Explain, briefly, why this result is *not* in conflict with the proposition that c is the speed of light observed in every locally inertial frame. (*4 marks*)

Question 16 This question concerns the evolution of a hypothetical homogeneous universe, for which Einstein's field equations lead to

$$\left(\frac{1}{R(t)}\frac{\mathrm{d}R(t)}{\mathrm{d}t}\right)^2 + \frac{kc^2}{R^2(t)} = \frac{8\pi G}{3c^2}\rho(t)$$

where $R(t)$ and $\rho(t)$ are the scale factor and total energy density, at time t, and k is the spatial curvature parameter. The total energy density receives a constant positive contribution ρ_Λ, from dark energy, a contribution $\rho_{\mathrm{matter}}(t)$, from matter, and a contribution $\rho_{\mathrm{radiation}}(t)$, from radiation.

(*a*) State the dependence of $\rho_{\mathrm{matter}}(t)$ and $\rho_{\mathrm{radiation}}(t)$ on the scale factor and give a brief explanation of the power of $R(t)$ in each case. (*4 marks*)

(*b*) Explain, briefly, why radiation is dominant at early times and hence verify the time dependence of the scale factor is $R(t) = At^{1/2}$, at small t, where A is a positive constant. (*4 marks*)

(*c*) Suppose that the Hubble parameter is $H_0 = 2.0 \times 10^{-18}\,\mathrm{s}^{-1}$ at time t_0. Use the constant

$$\frac{3c^2}{8\pi G} = 1.6 \times 10^{26}\,\mathrm{kg\,m}^{-1}$$

to determine the critical value ρ_0^{c} of the total energy density that would make this universe spatially flat. Make sure that you include appropriate units in your answer. (*2 marks*)

(*d*) Now suppose that the actual energy density ρ_0 at time t_0 is *less* than ρ_0^{c}. Will this universe continue to expand, or may it later contract? State two properties of geometrical figures that follow from the sign of the spatial curvature. (*3 marks*)

Question 17 Describe the ways in which the special and general theories of relativity help us to develop a fuller understanding of the propagation of light signals.

Your account should include brief explanations of:
the conflict between Galilean relativity and Maxwell's theory of light;
the influence of matter on the world-lines of light signals;
gravitational and cosmological redshifts. *(13 marks)*

Aim to use fewer than about 300 words. Marks will be given primarily for the relevance and clarity of your explanations.

Question 18 Describe how the spacetime curvature of the Universe was dominated by different sources at different epochs, according to current ideas.

Your account should include brief explanations of the effects of:
a large cosmological constant in a very early epoch of inflation;
the changing balance between the roles of radiation and matter, thereafter;
the likely balance of known matter, dark matter and dark energy in our present epoch. *(13 marks)*

Aim to use fewer than about 300 words. Marks will be given primarily for the relevance and clarity of your explanations.

[END OF QUESTION PAPER]

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PART I **AT THE END OF THE EXAMINATION, ATTACH THIS PART TO THE FRONT OF THE ANSWER BOOKS FOR PART II USING THE PAPER FASTENER PROVIDED.**

Examination No.								
Personal Identifier								

Please indicate on this grid which questions from Part II you have answered. (Details of Part I are *not* required.)

Part II			

Instructions for Part I

(*i*) Part I carries 48 per cent of the total marks, and you are advised to spend about **90 minutes** on it.

(*ii*) For each of the twelve questions, fill in the answer box(es) or space provided as indicated in the question.

(*iii*) Attempt as many questions as you can. There are **NO** penalties in Part I.

Question 1 (*4 marks*) Two particles, A and B, experience the same constant force **F** and are found to have the following position vectors (in metres) for times $t \geqslant 0$ (in seconds) in an inertial frame S:

$$\mathbf{x}_A(t) = (-6,\ 2t^2 + 1,\ t + 2)$$
$$\mathbf{x}_B(t) = (-4t - 2,\ t^2 + 2t,\ 4t - 1).$$

Use Newtonian mechanics to answer parts (*a*) to (*d*), assuming that SI units are used throughout.

For each of (*a*) to (*d*) select **ONE** item from the key for Q1.

(*a*) What is the numerical value of the time when the particles collide? ☐

(*b*) What is the numerical value of the particles' relative speed when they collide? ☐

(*c*) What is the numerical value obtained by dividing the mass of particle A by the mass of particle B? ☐

(*d*) Suppose that A and B combine to make a single particle, C. What is the numerical value of the magnitude of the acceleration of C when subjected to **F** in frame S? ☐

KEY for Q1

A 1/3

B 1/2

C 2/3

D 4/3

E 1

F 2

G 4

H 5

Question 2 (*4 marks*) Which **FOUR** of the statements in the key for Q2 are *true*, within the framework of Newtonian mechanics?

KEY for Q2

A The form of Newton's law of universal gravitation cannot be deduced from Kepler's second law alone.

B Observations of the gravitational force near a star S may be used to determine how the density of S varies with the distance from its centre.

C If a small planet P describes a circular orbit round a star S, then the mass of P may be determined by measuring the radius and period of the orbit.

D If a small planet P describes a circular orbit round a star S, then the mass of S may be determined by measuring the radius and period of the orbit.

E If two small planets, of equal mass, describe circular orbits round a star S, then the planet further from S has the greater magnitude of linear momentum.

F If two small planets, of equal mass, describe circular orbits round a star S, then the planet further from S has the greater magnitude of angular momentum.

G Conservation of linear momentum follows from the homogeneity of space.

H Conservation of angular momentum follows from the homogeneity of time.

Question 3 (*4 marks*) Observer O uses an inertial frame S.

Observer O_1 uses a frame with axes parallel to those of S, while its origin moves at constant velocity in S.

Observer O_2 uses a frame whose **1**-axis is identical to that of S at all times, while the other two axes are at a fixed angle relative to those of S.

Observer O_3 uses a frame whose 1-axis is identical to that of S at all times, while the other two axes are rotating at constant angular speed relative to those of S.

Suppose that O studies the motion of a pair of particles, A and B, and finds that the total momentum of the system is conserved and that the total energy is conserved when a particular form is assumed for the dependence of the potential energy on the distance between A and B.

Answer the following questions within the framework of Newtonian mechanics.

For each of (*a*) to (*d*), select **ONE** answer from the key for Q3.

(*a*) Who agrees with O about the value of the distance between A and B, at some particular time? ☐

(*b*) Who agrees with O about the values of the kinetic energies of A and B, at some particular time? ☐

(*c*) Who agrees with O about the values of the components of the momenta of A and B, at some particular time? ☐

(*d*) Who may use the same potential energy function that was used by O and correctly predict the motion using the force law that follows from energy conservation? ☐

KEY for Q3

A None of O_1, O_2, O_3

B Only O_1

C Only O_2

D Only O_3

E All but O_1

F All but O_2

G All but O_3

H All of O_1, O_2, O_3

Question 4 (*4 marks*) A proton is observed to move at a *constant* velocity **v** through a region containing constant uniform electric and magnetic fields **E** and **B**, respectively. Then an electron is released in this region, with an initial velocity **u**. Let u, v, E and B denote the non-zero magnitudes of the vectors **u**, **v**, **E** and **B**, respectively.

Which **FOUR** of the statements in the key for Q4 are *true*? | | | | |

KEY for Q4

A We can be sure that $v = E/B$.

B We cannot be sure that $v = E/B$.

C We can be sure that $v \geqslant E/B$.

D We can be sure that $v \leqslant E/B$.

E If $\mathbf{u} = \mathbf{v}$, then we can be sure that the electron will not accelerate.

F If $\mathbf{u} = -\mathbf{v}$, then we can be sure that the electron will accelerate.

G Since the electron is much less massive than the proton, we can be sure that the electron will accelerate.

H Since the electron is much less massive than the proton, we must require that $u \gg v$ if we wish to prevent the electron from accelerating.

Question 5 (*4 marks*) A long thin spaceship S of mass m has length L_0, when measured in its own rest frame. It carries a clock C whose period is T_0, when measured in its own rest frame. An inertial observer O finds that S is moving uniformly, with speed $v = \frac{3}{5}c$, in a direction parallel to its longest dimension.

For each of (*a*) to (*d*) select **ONE** item from the key for Q5.

(*a*) If O observes S to have length L, then what is the value of L/L_0? ☐

(*b*) If O observes C to have period T, then what is the value of T/T_0? ☐

(*c*) If O observes S to have relativistic momentum of magnitude p, then what is the value of p/mc? ☐

(*d*) If O observes S to have relativistic energy E, then what is the value of E/mc^2? ☐

KEY for Q5

A 0

B 3/5

C 3/4

D 4/5

E 1

F 5/4

G 4/3

H 5/3

Question 6 (*4 marks*) Which **FOUR** of the statements in the key for Q6 are *true*, within the framework of special relativity?

KEY for Q6

A If twin 1 remains at rest in an inertial frame and identical twin 2 leaves twin 1, takes a journey, and returns to twin 1, then twin 1 is *older* than twin 2 at the reunion.

B If twin 1 remains at rest in an inertial frame and identical twin 2 leaves twin 1, takes a journey, and returns to twin 1, then twin 1 is *younger* than twin 2 at the reunion.

C If a railway worker, W, for whom a tunnel is at rest, claims that a uniformly moving train is *longer* than the tunnel, then a passenger, on board the train, must agree with W.

D If a passenger, P, on board a train moving uniformly through a tunnel, claims that the train is *shorter* than the tunnel, then a railway worker, for whom the tunnel is at rest, must agree with P.

E If the invariant interval between two events is *space*-like, then neither event can be a possible cause of the other.

F If the invariant interval between two events is *time*-like, then neither event can be a possible cause of the other.

G If one event causes another, then the invariant interval between the events *must* be space-like.

H If one event causes another, then the invariant interval between the events *must* be time-like.

Question 7 (*4 marks*) This question concerns 4 redshifts z_1, z_2, z_3 and z_4 with *very* different sizes.

For each of (*a*) to (*d*), select the **ONE** answer from the key for Q7 that is closest to your very rough estimate of the redshift in question.

(*a*) The special-relativistic redshift z_1 is detected by an inertial observer O, in the absence of gravity, in the light from a source that is moving directly away from O at a uniform speed of $300\,\mathrm{km\,s^{-1}}$. What is the order of magnitude of z_1? ☐

(*b*) The gravitational redshift z_2 is detected by a pair of physicists at the top of a tower at Harvard university, when observing γ-rays from a source at the bottom of the tower. What is the order of magnitude of z_2? ☐

(*c*) The gravitational redshift z_3 is detected by an astronaut who maintains fixed spatial coordinates, at a very great distance from the Earth, and happens to detect a γ-ray coming from Harvard. What is the order of magnitude of z_3? ☐

(*d*) The cosmological redshift z_4 is detected by a pair of telephone scientists, studying radio noise at microwave wavelengths. It is later interpreted as the redshift of a thermal spectrum during the entirety of the cosmic epoch that has elapsed since this radiation acquired its thermal character by interacting with matter. What is the order of magnitude of z_4? ☐

KEY for Q7

A 10^{-21}

B 10^{-15}

C 10^{-9}

D 10^{-3}

E 1

F 10^{3}

G 10^{6}

H 10^{9}

Question 8 (*4 marks*) Which **FOUR** of the statements in the key for Q8 are *true*?

KEY for Q8

A A spacetime geodesic connecting a pair of events minimizes the spatial length of the world-line connecting the events.

B Spacetime may be curved even when space is flat.

C Every region of spacetime that contains no sources of curvature is necessarily flat.

D Einstein's field equations relate all the components of the Riemann curvature directly to source terms.

E Einstein's field equations relate all the components of the Ricci curvature directly to source terms.

F The inclusion of a cosmological constant in Einstein's field equations is very recent; Einstein never considered such a term.

G The fall of an apple near the Earth's surface may be predicted using the Schwarzschild metric, with no reference to gravitational force.

H The fate of our Universe may be predicted using the Robertson–Walker metric, if we are given the Hubble parameter and the energy densities of matter, radiation and dark energy.

Question 9 (*4 marks*) Which **FOUR** of the items in the key for Q9 are *true* statements about black holes?

KEY for Q9

A A falling object takes a finite time to reach the event horizon, according to an observer who maintains fixed spatial coordinates at a finite distance from a black hole.

B A falling object takes a finite time to reach the event horizon, according to a clock falling with that object.

C A falling object takes a finite time to reach the central singularity, according to a clock falling with that object.

D Tidal effects near the event horizon of a black hole are far more severe for galactic-centre black holes than for stellar black holes.

E The Schwarzschild metric can be used to describe spacetime exterior to the event horizon of a black hole that carries angular momentum.

F A black hole may carry charge.

G When a black hole increases in mass, the area of its event horizon increases.

H No black hole can decrease in mass.

Question 10 (*4 marks*) This question concerns 4 of the many scientists who have made distinctive contributions to our understanding of the Universe.

For each of (*a*) to (*d*) select **ONE** item from the key for Q10.

(*a*) Who first taught us that all motion is relative? ☐

(*b*) Who gave us the equations of electromagnetism that lead from the relativity of motion to the relativity of simultaneity asserted in the special theory of relativity? ☐

(*c*) Who first applied the general theory of relativity to link the evolution of our Universe to the energy density and pressure of its contents? ☐

(*d*) Who predicted that we might observe on Earth signals from an epoch when the energy density of our Universe was a billion times greater than it is now? ☐

KEY for Q10

A Einstein

B Friedmann

C Galileo

D Gamow

E Maxwell

F Newton

G Penzias

H Robertson

Question 11 (*4 marks*) The Robertson–Walker metric takes the form

$$(\Delta S)^2 = c^2(\Delta t)^2 - R^2(t)\left[\frac{(\Delta\sigma)^2}{1-k\sigma^2} + \sigma^2(\Delta\theta)^2 + \sigma^2\sin^2\theta(\Delta\phi)^2\right].$$

In this question, assume that the cosmological constant is not negative.

Select from the key for Q11 the **FOUR** statements that are *true*.

KEY for Q11

A Our own cluster of galaxies *must* be chosen as the origin of spatial coordinates.

B If $k < 0$, then the Universe must continue to expand.

C If $k = 0$, then space is flat.

D If $k > 0$, then the distance from the origin of a cluster of galaxies with comoving coordinate σ is *greater* than $R(t)\sigma$ at time t.

E If $k > 0$, then the distance from the origin of a cluster of galaxies with comoving coordinate σ is *less* than $R(t)\sigma$ at time t.

F If $k > 0$, then a triangle connecting three widely separated clusters of galaxies will have angles that sum to *more* than 180°.

G If $k > 0$, then a triangle connecting three widely separated clusters of galaxies will have angles that sum to *less* than 180°.

H If the curvature of space is zero and the pressure $p(t_0)$ is known at time t_0, then the value of the Hubble parameter $H(t_0)$ at time t_0 may be computed, without knowing the energy density $\rho(t_0)$.

Question 12 (*4 marks*) This question concerns current suppositions about the possible ingredients of our Universe.

For each of (*a*) to (*d*) select **ONE** item from the key for Q12.

(*a*) What item is supposed to make the *most* important contribution to the present spacetime curvature of our Universe? ☐

(*b*) What item is supposed to make the *second* most important contribution to the present spacetime curvature of our Universe? ☐

(*c*) What item do we chiefly observe in order to estimate the critical value that the energy density must take if our Universe is spatially flat? ☐

(*d*) What item is supposed to have been important before inflation, but negligible thereafter? ☐

KEY for Q12

A bright stars in galaxies

B brown dwarfs

C cold dark matter

D dark energy

E high-energy protons

F hot dark matter

G magnetic monopoles

H primordial ^{6}Li nuclei