S357/U

The Open University

Third Level Course Examination 2002

Space, Time and Cosmology

Thursday 17th October 2002   2.30pm–5.30pm

Time allowed: 3 hours

This paper is divided into **TWO** parts, I and II. Part I carries 48 per cent of the total marks, and Part II carries 52 per cent. You are advised to spend roughly equal time on each part. Remember to allow time for reading the question paper first, and your answers afterwards. You will not be given extra time to do this.

You should answer **ALL** the questions in Part I, and **FOUR** questions from Part II.

Record your answers to Part I in the spaces provided in the green question and answer booklet inserted in this question paper. Answer **each** question attempted from Part II in a **separate** answer book (i.e. use **four** answer books in all). *It is most important that you indicate in the box provided on the front of each answer book the number of the question which you have answered in that book.*

At the end of the examination

1. Make sure that you have written your personal identifier and examination number on Part I of the question paper and on **each** answer book used. **Failure to do so will mean that your work cannot be identified.**
2. Complete the grid on the front of the enclosed green booklet containing Part I.
3. Put your signed desk record on top of Part I of the question paper and the answer books in which you have answered questions from Part II, and fix them together with the paper fastener provided.

Note: Part I of this paper is provided as a separate insert.

PART II (*i*) *Part II carries 52 per cent of the total marks, and you are advised to spend about* **90 minutes** on it.

(*ii*) *Attempt* **FOUR** *questions only.*

(*iii*) *All answers carry thirteen marks, and where questions are divided into more than one section (labelled (a), (b), (c), etc.), the marks allocated to each section are indicated.*

(*iv*) *Write your answer to* **each** *question from Part II in a* **separate** *answer book. You should therefore submit* **FOUR** *answer books for this part.*

Question 13 This question concerns Newton's third law in a two-particle system.

At time t, particle 1, with mass m_1, has position $\mathbf{x}_1$, velocity $\mathbf{v}_1$, momentum $\mathbf{p}_1$, and acceleration $\mathbf{a}_1$. It is acted on by a force $\mathbf{F}_1$ exerted by particle 2.

Similarly, particle 2, with mass m_2, has position $\mathbf{x}_2$, velocity $\mathbf{v}_2$, momentum $\mathbf{p}_2$, and acceleration $\mathbf{a}_2$. It is acted on by a force $\mathbf{F}_2$ exerted by particle 1.

(*a*) Write down two relations between the four vectors $\mathbf{v}_1$, $\mathbf{p}_1$, $\mathbf{a}_1$ and $\mathbf{F}_1$, within the framework of Newtonian mechanics. (*2 marks*)

(*b*) State how one of these relations is modified in the special theory of relativity, and why the other can be erased completely in the case of a gravitational interaction described within the framework of general relativity. (*2 marks*)

(*c*) Write down a relation between the vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ that is asserted by that part of Newton's third law which ensures conservation of total linear momentum. Derive from it the constancy of $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$. (*3 marks*)

(*d*) Write down a further condition which completes the statement of Newton's third law. Define the total angular momentum of the system and show that it is conserved. (*4 marks*)

(*e*) In the interactions between charged particles implied by Einstein's paper of 1905 on the electrodynamics of moving bodies, the sum of the momenta of the particles is *not*, in general, conserved (even after taking account of Einstein's different definition of the momentum of a particle).

What phenomenon reconciles this apparent discrepancy with the conservation of *total* momentum?

What feature of Einstein's field equations of general relativity recognizes the existence of momentum that is not carried by massive particles? (*2 marks*)

S357/U

The Open University

PART I **AT THE END OF THE EXAMINATION, ATTACH THIS PART TO THE FRONT OF THE ANSWER BOOKS FOR PART II USING THE PAPER FASTENER PROVIDED.**

Examination No.								
Personal Identifier								

Please indicate on this grid which questions from Part II you have answered. (Details of Part I are *not* required.)

Part II			

Instructions for Part I

(*i*) Part I carries 48 per cent of the total marks, and you are advised to spend about **90 minutes** on it.

(*ii*) For each of the twelve questions, fill in the answer box(es) or space provided as indicated in the question.

(*iii*) Attempt as many questions as you can. There are **NO** penalties in Part I.

Question 1 (*4 marks*) Two particles, A and B, move under the influence of the same constant force **F** in an inertial frame S. At time t (in seconds) their position vectors (in metres) are

$$\mathbf{x}_{\mathrm{A}} = (6t^2 + 4, 5t, 4t + 1), \quad \mathbf{x}_{\mathrm{B}} = (2t^2 + 8t, 2t + 3, 5).$$

For each of (*a*) to (*d*), select **ONE** answer from the key for Q1.

(*a*) What is the ratio of the mass of B to that of A? ☐

(*b*) At what value of t (in seconds) do the particles collide? ☐

(*c*) What is the speed (in $\mathrm{m\,s^{-1}}$) of A relative to B when the particles collide? ☐

(*d*) Suppose that A and B combine to make a single particle, C. What is the magnitude (in $\mathrm{m\,s^{-2}}$) of the acceleration of C when subjected to **F** in frame S? ☐

KEY for Q1

A 1/5

B 1/4

C 1/3

D 1/2

E 1

F 2

G 3

H 4

I 5

Question 2 (*4 marks*) This question concerns 4 observers, who assign the same times to events.

Observer O uses an inertial frame S and assigns spatial coordinates (x, y, z) to an arbitrary event E at time t.

Observer P uses a coordinate frame obtained from S by a rotation of axes and assigns spatial coordinates $((x - y)/\sqrt{2}, (x + y)/\sqrt{2}, z)$ to event E.

Observer Q is in motion relative to S and assigns spatial coordinates $(x, y, z - vt)$ to event E, where v is a positive constant.

Observer R uses the same axes as P, moves relative to S, and assigns spatial coordinates $((x - y)/\sqrt{2}, (x + y)/\sqrt{2}, z - vt)$ to event E.

Answer the following questions within the framework of *Galilean* relativity, in which time is supposed to be universal.

For each of (a) to (d), select **ONE** answer from the key for Q2.

(a) Who agrees with O about the distances between particles? ☐

(b) Who agrees with O about the velocity vectors of particles? ☐

(c) Who agrees with O about the acceleration vectors of particles? ☐

(d) Who agrees with O about the magnitudes of the acceleration of particles? ☐

KEY for Q2

A None of P, Q, R

B Only P

C Only Q

D Only R

E All but P

F All but Q

G All but R

H All of P, Q, R

Question 3 (*4 marks*) Answer the following questions within the framework of Newtonian mechanics.

For each of (a) to (d), select **ONE** answer from the key for Q3.

(a) Which item is most directly related to the homogeneity of time? ☐

(b) Which item is most directly related to the homogeneity of space? ☐

(c) Which item is most directly related to the isotropy of space? ☐

(d) Which item is most directly related to the inverse square law of universal gravitation? ☐

KEY for Q3

A Newton's second law

B Kepler's second law (equal areas swept out in equal times)

C Kepler's third law (periods of orbits related to their radii)

D Conservation of mass

E Conservation of energy

F Conservation of linear momentum

Question 4 (*3 marks*) The rate of change of the momentum $\mathbf{p}$ of a particle with velocity $\mathbf{v}$ and charge q at a point in an inertial frame S where the electric and magnetic fields are $\mathbf{E}$ and $\mathbf{B}$, respectively, is given by

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}).$$

The theory of special relativity asserts that one may use an equation of the same *form:*

$$\frac{d\mathbf{p}'}{dt'} = q'(\mathbf{E}' + \mathbf{v}'\times\mathbf{B}'),$$

in an inertial frame S′ moving uniformly with respect to S.

Let m be the mass of the particle and c the speed of light.

Select from the key for Q4 the **THREE** items that are *true* within the framework of special relativity. ☐☐☐

KEY for Q4

A $t' = t$

B $q' = q$

C $\mathbf{E}' = \mathbf{E}$

D $\mathbf{B}' = \mathbf{B}$

E $\mathbf{p} = m\mathbf{v}$

F $\mathbf{p}' = m\mathbf{v}'$

G $\mathbf{p} = m\mathbf{v}/\sqrt{1 - |\mathbf{v}|^2/c^2}$

H $\mathbf{p}' = m\mathbf{v}'/\sqrt{1 - |\mathbf{v}'|^2/c^2}$

Question 5 (*4 marks*) This question concerns three events that occur on the x-axis of an inertial frame. Their spacetime coordinates, in the conventional notation (ct, x) of special relativity, are specified by $\mathcal{E}_1 = (-2a, a)$, $\mathcal{E}_2 = (0, 2a)$ and $\mathcal{E}_3 = (2a, 0)$, where a is a positive distance.

Select from the key for Q5 the **FOUR** statements that are *true* within the framework of special relativity.

KEY for Q5

A There is an inertial frame in which $\mathcal{E}_1$ and $\mathcal{E}_2$ occur at the same position.

B There is an inertial frame in which $\mathcal{E}_1$ and $\mathcal{E}_3$ occur at the same position.

C There is an inertial frame in which $\mathcal{E}_2$ and $\mathcal{E}_3$ occur at the same position.

D There is an inertial frame in which $\mathcal{E}_1$ and $\mathcal{E}_2$ occur at the same time.

E There is an inertial frame in which $\mathcal{E}_1$ and $\mathcal{E}_3$ occur at the same time.

F There is an inertial frame in which $\mathcal{E}_2$ and $\mathcal{E}_3$ occur at the same time.

G $\mathcal{E}_1$ is a possible cause of $\mathcal{E}_2$.

H $\mathcal{E}_1$ is a possible cause of $\mathcal{E}_3$.

I $\mathcal{E}_2$ is a possible cause of $\mathcal{E}_3$.

Question 6 (*4 marks*) In an inertial frame S, a particle A and an antiparticle B, each of mass m, move directly towards each other. Each has speed v. Their momenta have opposite directions and the same magnitude p. The head-on collision annihilates A and B and produces only two photons, each with energy $\frac{5}{3}mc^2$.

For each of (*a*) and (*b*), select **ONE** answer from the key for Q6.

(*a*) What is the value of v/c ? ☐

(*b*) What is the value of p/mc ? ☐

KEY for Q6

A 0

B 3/5

C 3/4

D 4/5

E 1

F 5/4

G 4/3

H 5/3

Question 7 (*4 marks*) This question concerns the behaviour of two identical clocks, X and Y, each of which participates in two experiments.

(*a*) The first experiment is conducted in an inertial frame, in the absence of gravity.

X remains at rest at point P. After synchronization of X and Y at P, Y is sent on a round trip. On its return to P, the readings of X and Y are compared.

Select from the key for Q7 the **TWO** items that apply to the first experiment.

(*b*) The second experiment is conducted near to a massive spherically symmetric distribution of matter.

X is held at point Q, with fixed Schwarzschild coordinates. After synchronization of X and Y at Q, Y is sent on a round trip, by throwing it in the direction of increasing radial coordinate. On its return to Q, the readings of X and Y are compared.

Select from the key for Q7 the **TWO** items that apply to the second experiment.

KEY for Q7

A On its return, Y reads more than X.

B On its return, Y reads the same as X.

C On its return, Y reads less than X.

D The world-line of X is a spacetime geodesic.

E During its trip, the world-line of Y is a spacetime geodesic.

F There is complete symmetry between X and Y.

Question 8 (*4 marks*) Select from the key for Q8 the **FOUR** items that are most essential to the understanding of general relativity as a metric theory of motion under gravity.

KEY for Q8

A The existence of locally inertial frames

B The generation of Ricci curvature by matter and radiation

C The geodesic principle

D Newton's second law

E Newton's law of universal gravitation

F Hawking radiation

G The vanishing of the cosmological constant

H The vanishing of the proper time interval between events on the world-line of a light signal

Question 9 (*4 marks*) Select from the key for Q9 the **FOUR** items that provide experimental evidence confirming distinctive predictions of the general theory of relativity.

KEY for Q9

A The annihilation of particles by antiparticles

B The bending of starlight observed during a solar eclipse

C The changes in orbital period of the Hulse–Taylor binary

D The existence of pulsars

E The Kennedy–Thorndike experiment

F The Mössbauer effect

G The Pound–Rebka experiment

H The time delay of radar signals passing close to the Sun

Question 10 (*4 marks*) The metric describing the large-scale spacetime structure of the universe takes the form

$$(\Delta S)^2 = c^2(\Delta t)^2 - R^2(t)\left[\frac{(\Delta\sigma)^2}{1-k\sigma^2} + \sigma^2(\Delta\theta)^2 + \sigma^2\sin^2\theta(\Delta\phi)^2\right]$$

where k is a constant, which is conventionally assigned one of three values: $k = 1$, $k = 0$, or $k = -1$.

For each of (*a*) to (*d*), select **ONE** answer from the key for Q10.

(*a*) For which value(s) of k is the circumference of a large circle, with constant comoving coordinate σ, less than $2\pi R(t)\sigma$? ☐

(*b*) For which value(s) of k is the sum of the angles of a large triangle, formed by geodesics, less than 180°? ☐

(*c*) For which value(s) of k may the universe cease to expand? ☐

(*d*) For which value(s) of k is *spacetime* flat? ☐

KEY for Q10

A For none of the three k values

B Only for $k = 1$

C For $k = 1$ and for $k = 0$ but not for $k = -1$

D Only for $k = 0$

E For $k = 0$ and for $k = -1$ but not for $k = 1$

F Only for $k = -1$

G For all three k values

Question 11 (*4 marks*) A light signal from a very distant cluster of galaxies was emitted at time t_1 with wavelength λ_1. Much later it is received on Earth, at time t_2, with a longer wavelength λ_2, having suffered a cosmological redshift $z = \lambda_2/\lambda_1 - 1$.

For each of (*a*) to (*d*), select **ONE** answer from the key for Q11.

(*a*) What is the ratio of the large-scale density of matter at time t_1 to that at time t_2? ☐

(*b*) What is the ratio of the temperature of the cosmic microwave background radiation at time t_1 to that at time t_2? ☐

(*c*) What is the ratio of the energy density of the cosmic microwave background radiation at time t_1 to that at time t_2? ☐

(*d*) Assuming that the universe is not spatially flat, what is the ratio of the large-scale spatial curvature at time t_1 to that at time t_2? ☐

KEY for Q11

A 1

B z

C $1 + z$

D z^2

E $(1 + z)^2$

F z^3

G $(1 + z)^3$

H z^4

I $(1 + z)^4$

Question 12 (*5 marks*) In each part, fill in the blanks with appropriate letters from the key for Q12. Use no letter more than once.

(*a*) The Schwarzschild metric for a spherically symmetric black hole predicts the existence of two notable features: the [ ], from inside which there is no escape, and the [ ], where light signals may describe circular orbits.

If the black hole is rotating, the appropriate solution is given by the [ ], which predicts the existence of a third feature: the [ ], from which escape is possible, but within which nothing may remain at rest.

(*b*) The large-scale spacetime structure of a universe satisfying the cosmological principle is described by the Robertson–Walker metric. Its scale factor evolves with time in accordance with the [ ], into which one may introduce a [ ]. Had such a term been present in the early universe, it might have led to [ ].

This possibility has been invoked in attempts to explain three notable puzzles: the present universe is close to having [ ], the [ ] is highly isotropic, and we have not detected relics such as a [ ].

KEY for Q12

A Cosmological constant

B Critical density

C Ergosphere

D Event horizon

E Exponential inflation

F Friedmann equations

G Kerr metric

H Magnetic monopole

I Microwave background

J Photon sphere

Question 14 (a) Write down the four-dimensional spacetime metric of special relativity and derive from it the dependence on v of $\gamma(v) \equiv \Delta t/\Delta\tau$, where $\Delta\tau$ is the proper time elapsed on a clock moving with speed v during a coordinate time interval Δt. *(3 marks)*

(b) Now consider uniform motion of a particle of mass m at speed v in the positive x^1-direction. Let its relativistic energy, $E = p^0 c$, and momentum p^1, be defined by

$$p^0 = m\frac{\mathrm{d}x^0}{\mathrm{d}\tau}$$
$$p^1 = m\frac{\mathrm{d}x^1}{\mathrm{d}\tau}$$

with $x^0 = ct$. Show that $(p^0)^2 - (p^1)^2 = (mc)^2$. *(3 marks)*

(c) Define the relativistic kinetic energy to be the relativistic energy minus the energy at rest. Show that it is greater than $\frac{1}{2}mv^2$, by using the inequality

$$1/\sqrt{1-x} > 1 + \frac{1}{2}x,$$

which holds when $1 > x > 0$. *(3 marks)*

(d) The relativistic energy of an electron at rest is 511 keV. Suppose that an electron is accelerated from rest through a potential difference of 511 kilovolts. Calculate its final speed, taking care to show all your working and to avoid any non-relativistic approximation. *(4 marks)*

Question 15 This question concerns radial motion of a test particle of mass m in the spacetime exterior to a spherically symmetric distribution of matter of mass $M \gg m$, with centre O. The particle is released from rest at an (effectively) infinite distance from O and falls radially towards O.

(*a*) In the approximate version of this problem afforded by Newtonian mechanics, one may use energy conservation with a potential energy function $U = -GMm/r$.

Show that this leads to a speed $\sqrt{2GM/r}$ at a distance r from O. (*3 marks*)

(*b*) In general relativity, one may use the Schwarzschild metric

$$(c\,\Delta\tau)^2 = \left(1 - \frac{k}{r}\right)(c\,\Delta t)^2 - \left\{\frac{(\Delta r)^2}{1 - \dfrac{k}{r}} + r^2(\Delta\theta)^2 + r^2\sin^2\theta\,(\Delta\phi)^2\right\}$$

with $k = 2GM/c^2$.

Explain, briefly, what is meant by:
loss of metrical significance of the radial coordinate difference Δr;
loss of metrical significance of the time coordinate difference Δt;
invariance of the proper time interval $\Delta\tau$. (*3 marks*)

(*c*) Radial motion in this metric conserves the value of

$$N = \left(1 - \frac{k}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}.$$

Show that

$$\left(1 - \frac{k}{r}\right) = N^2 - \left(\frac{1}{c}\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2.$$

(*4 marks*)

(*d*) Show that the initial conditions imply that $N^2 = 1$ and hence that the radial coordinate decreases with *proper* time at a rate $\sqrt{2GM/r}$. (*3 marks*)

Question 16 This question concerns a hypothetical homogeneous universe, for which Einstein's field equations lead to

$$\left(\frac{1}{R(t)}\frac{\mathrm{d}R(t)}{\mathrm{d}t}\right)^2 + \frac{kc^2}{R^2(t)} = \frac{8\pi G}{3c^2}\rho(t)$$

where $R(t)$ and $\rho(t)$ are respectively the scale factor and energy density at universal time t, k is the spatial curvature parameter, and

$$\frac{3c^2}{8\pi G} = 1.6 \times 10^{26}\,\mathrm{kg\,m^{-1}}.$$

The energy density receives contributions from both matter and radiation.

(*a*) At a certain time t_{eq} the energy densities of matter and radiation were equal, each having the value $5.0\,\mathrm{J\,m^{-3}}$, and the Hubble parameter was $2.5 \times 10^{-13}\,\mathrm{s^{-1}}$. Show that this universe is spatially flat. *(2 marks)*

(*b*) State how each of the following depends upon the scale factor $R(t)$:
the energy density of matter;
the temperature of radiation;
the energy density of radiation.

In each case, give a brief explanation for the power of the scale factor. *(3 marks)*

(*c*) Show that the Hubble parameter $H(t)$ and the temperature of radiation $T(t)$ are related, at time $t > t_{\mathrm{eq}}$, by the equation

$$2\left(\frac{H(t)}{H(t_{\mathrm{eq}})}\right)^2 = \left(\frac{T(t)}{T(t_{\mathrm{eq}})}\right)^3 + \left(\frac{T(t)}{T(t_{\mathrm{eq}})}\right)^4.$$

(4 marks)

(*d*) Suppose that the temperature was 9000 K when the densities of matter and radiation were equal. Much later, when intelligent life has evolved in this universe, cosmologists observe a Hubble parameter of $1.0 \times 10^{-18}\,\mathrm{s^{-1}}$. What temperature will they find for the cosmic microwave background radiation? *(4 marks)*

Question 17 Describe the ways in which the special and general theories of relativity developed a fuller understanding of the propagation of light signals.

Your account should include brief explanations of:
the conflict between Galilean relativity and Maxwell's theory of light;
the influence of matter on the world-lines of light signals;
gravitational and cosmological redshifts. *(13 marks)*

Aim to use fewer than about 300 words. Marks will be given primarily for the relevance and clarity of your explanation.

Question 18 Explain the sense in which (so it is said) 'a black hole has no hair'.

Your discussion should include brief explanations of:
how a black hole acts as a 'hole';
why it is almost, but not quite, 'black';
loss of detailed information (the 'hair' referred to above) in its formation. *(13 marks)*

Aim to use fewer than about 300 words. Marks will be given primarily for the relevance and clarity of your explanation.

[END OF QUESTION PAPER]