S357/T

The Open University

Third Level Course Examination 2001

Space, Time and Cosmology

Thursday, 18th October, 2001 10.00 am–1.00 pm

Time allowed: 3 hours

This paper is divided into **TWO** parts, I and II. Part I carries 48 per cent of the total marks, and Part II carries 52 per cent. You are advised to spend roughly equal time on each part. Remember to allow time for reading the question paper first, and your answers afterwards. You will not be given extra time to do this.

You should answer **ALL** the questions in Part I, and **FOUR** questions from Part II.

Record your answers to Part I in the spaces provided in the green question and answer booklet inserted in this question paper. Answer **each** question attempted from Part II in a **separate** answer book (i.e. use **four** answer books in all). *It is most important that you indicate in the box provided on the front of each answer book the number of the question which you have answered in that book.*

At the end of the examination

1. Make sure that you have written your personal identifier and examination number on Part I of the question paper and on **each** answer book used. **Failure to do so will mean that your work cannot be identified.**
2. Complete the grid on the front of the enclosed green booklet containing Part I.
3. Put your signed desk record on top of Part I of the question paper and the answer books in which you have answered questions from Part II, and fix them together with the paper fastener provided.

Note: Part I of this paper is provided as a separate insert.

PART II *(i) Part II carries 52 per cent of the total marks, and you are advised to spend about* **90 minutes** on it.

(ii) Attempt **FOUR** *questions only.*

(iii) All answers carry thirteen marks, and where questions are divided into more than one section (labelled (a), (b), (c), etc.), the marks allocated to each section are indicated.

(iv) Write your answer to **each** *question from Part II in a* **separate** *answer book. You should therefore submit* **FOUR** *answer books for this part.*

Question 13 An isolated system consists of three interacting particles: A, B and C. Suppose that A and B are identical and have mass m, while C has mass M. In an inertial frame S, the initial positions (in metres) and velocities (in metres per second) are

$$\mathbf{x}_A(0) = (0, 1, 0) \quad \mathbf{v}_A(0) = (0, 2, -1)$$
$$\mathbf{x}_B(0) = (0, -1, 0) \quad \mathbf{v}_B(0) = (0, -2, -1)$$
$$\mathbf{x}_C(0) = (0, 0, 2.5) \quad \mathbf{v}_C(0) = (0, 0, 1).$$

(a) Write down these initial conditions in frame S′ obtained by a rotation through 180° about the **3**-axis of S. Give a reason why S′ is also an inertial frame. *(2 marks)*

(b) "Suppose inertial observers in isolated frames S and S′...". Complete the statement of the principle of relativity in Newtonian physics, making appropriate reference to identical particles and initial positions and velocities. (Any wording which clearly includes the essential points is acceptable.) *(2 marks)*

(c) Making use of your answer to part *(b)*, argue from the Newtonian principle of relativity that C cannot leave the **3**-axis of S. *(2 marks)*

(d) At a later time t_1, it is found that

$$\mathbf{x}_B(t_1) = (-1, 0, 0.5) \quad \mathbf{v}_B(t_1) = (u, 0, 5)$$
$$\mathbf{x}_C(t_1) = (0, 0, 2) \quad \mathbf{v}_C(t_1) = (0, 0, -5)$$

where u is non-zero.

Use your answers to parts *(a)* and *(b)* to write down $\mathbf{x}_A(t_1)$ and $\mathbf{v}_A(t_1)$. *(1 mark)*

(e) Use the conservation of linear momentum to find the mass M of C in terms of m. *(2 marks)*

(f) What conservation principles follow from *(i)* the isotropy of space, *(ii)* the homogeneity of time? *(1 mark)*

(g) Explain why neither principle is of help in determining the unknown value u. *(3 marks)*

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PART I **AT THE END OF THE EXAMINATION, ATTACH THIS PART TO THE FRONT OF THE ANSWER BOOKS FOR PART II USING THE PAPER FASTENER PROVIDED.**

Examination No.								
Personal Identifier								

Please indicate on this grid which questions from Part II you have answered. (Details of Part I are *not* required.)

Part II			

Instructions for Part I

(*i*) Part I carries 48 per cent of the total marks, and you are advised to spend about **90 minutes** on it.

(*ii*) For each of the twelve questions, fill in the answer box(es) or space provided as indicated in the question.

(*iii*) Attempt as many questions as you can. There are **NO** penalties in Part I.

Question 1 (*4 marks*) Two particles, A and B, move under the influence of the same constant force **F** in an inertial frame S. At time t (in seconds) their position vectors (in metres) are

$$\mathbf{x}_A = (t + 4, 2t^2 + 1, 4), \quad \mathbf{x}_B = (4t + 1, t^2 + 2t, 4t).$$

For each of (*a*) to (*d*), select **ONE** answer from the key for Q1.

(*a*) What is the ratio of the mass of B to that of A? ☐

(*b*) At what value of t (in seconds) do the particles collide? ☐

(*c*) What is the speed (in $\mathrm{m\,s^{-1}}$) of A relative to B when the particles collide? ☐

(*d*) Suppose that A and B combine to make a single particle, C. What is the magnitude (in $\mathrm{m\,s^{-2}}$) of the acceleration of C when subjected to **F** in frame S? ☐

KEY for Q1

A 1/4

B 1/3

C 1/2

D 2/3

E 1

F 4/3

G 2

H 3

I 4

J 5

Question 2 (4 marks) Observer O uses an inertial frame S.

Observer O_1 uses a frame with axes parallel to those of S, while its origin moves at constant velocity in S.

Observer O_2 uses a frame with one axis identical to that of S, while the other two axes are at a fixed angle relative to those of S.

Observer O_3 uses a frame with one axis identical to that of S, while the other two axes are rotating at constant angular speed relative to those of S.

Answer the following questions within the framework of *Galilean* relativity.

For each of (*a*) to (*d*), select **ONE** answer from the key for Q2.

(*a*) Who agrees with O about the distances between particles? ☐

(*b*) Who agrees with O about the velocity vectors of particles? ☐

(*c*) Who agrees with O about the acceleration vectors of particles? ☐

(*d*) Who agrees with O about the magnitudes of the acceleration of particles? ☐

KEY for Q2

A none of O_1, O_2, O_3

B only O_1

C only O_2

D only O_3

E all but O_1

F all but O_2

G all but O_3

H all of O_1, O_2, O_3

Question 3 (*4 marks*) A hypothetical star S has two planets, A and B. Planet A moves in a circular orbit of radius r. Planet B moves in a highly elliptical orbit, with its distance from S varying between a minimum of r and a maximum of $49r$. The planes of the orbits are such that the gravitational interaction between A and B may be ignored.

For each of (*a*) to (*c*), determine the stated ratio and select **ONE** answer from the key for Q3.

(*a*) The ratio of the maximum and minimum rates at which area is swept out by the position vector of B relative to S. ☐

(*b*) The ratio of the period of the orbit of B to that of A. ☐

(*c*) The maximum ratio of the magnitude of acceleration of A to that of B. ☐

KEY for Q3

A 1

B 5

C 7

D 5^2

E 7^2

F 5^3

G 7^3

H 5^4

I 7^4

J 5^5

Question 4 (*5 marks*) Before Einstein's 1905 paper on the electrodynamics of moving bodies, motion of a charged particle was described by the equation

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}).$$

Select from the key for Q4 the **FIVE** statements that are *true* within the framework of special relativity.

KEY for Q4

A Maxwell's description of electric fields remains valid.

B Maxwell's description of electric fields has been modified.

C Maxwell's description of magnetic fields remains valid.

D Maxwell's description of magnetic fields has been modified.

E The invariance of charge has been abandoned.

F The invariance of time differences has been abandoned.

G The Newtonian definition of momentum has been modified.

H The equation of motion above has been modified.

I As written, the equation of motion remains valid in all inertial frames.

Question 5 (*4 marks*) In an inertial frame S, three events, $\mathcal{E}_A$, $\mathcal{E}_B$ and $\mathcal{E}_C$, occur on the x-axis at times $t_A = -T$, $t_B = 0$ and $t_C = 3T$, where $T > 0$. Their x-coordinates are $x_A = -3cT$, $x_B = -cT$ and $x_C = cT$.

Select from the key for Q5 the **FOUR** statements that are *true* within the framework of special relativity.

KEY for Q5

A There is an inertial frame in which $\mathcal{E}_A$ and $\mathcal{E}_B$ occur at the same position.

B There is an inertial frame in which $\mathcal{E}_A$ and $\mathcal{E}_C$ occur at the same position.

C There is an inertial frame in which $\mathcal{E}_B$ and $\mathcal{E}_C$ occur at the same position.

D There is an inertial frame in which $\mathcal{E}_A$ and $\mathcal{E}_B$ occur at the same time.

E There is an inertial frame in which $\mathcal{E}_A$ and $\mathcal{E}_C$ occur at the same time.

F There is an inertial frame in which $\mathcal{E}_B$ and $\mathcal{E}_C$ occur at the same time.

G $\mathcal{E}_A$ is a possible cause of $\mathcal{E}_B$.

H $\mathcal{E}_A$ is a possible cause of $\mathcal{E}_C$.

I $\mathcal{E}_B$ is a possible cause of $\mathcal{E}_C$.

Question 6 (*4 marks*) In an inertial frame S, a particle A and an antiparticle B, each of mass m, move directly towards each other. Each has speed v. Their momenta have opposite directions and the same magnitude p. The head-on collision annihilates A and B and produces only electromagnetic radiation, of total energy $E = \frac{5}{2}mc^2$.

For each of (*a*) and (*b*), select **ONE** answer from the key for Q6.

(*a*) What is the value of v/c ? ☐

(*b*) What is the value of p/mc ? ☐

KEY for Q6

A 0

B 3/5

C 3/4

D 4/5

E 1

F 5/4

G 4/3

H 5/3

Question 7 (*4 marks*) Consider two very precise identical clocks, A and B, sitting on a shelf, which resists the Earth's gravity. They are synchronized. A is left on the shelf. B is thrown vertically upwards, caught and put back on the shelf. The readings of A and B are compared.

Select from the key for Q7 the **FOUR** statements that are *true*. | | | | |

KEY for Q7

A In an inertial frame, moving clocks are observed to run slow.

B In an inertial frame, moving clocks are observed to run fast.

C If B were at a fixed height above A, it would run more slowly than A.

D If B were at a fixed height above A, it would run faster than A.

E The final readings agree, because the effects of motion and gravity cancel.

F B reads more than A, because B spent time in free fall.

G A reads more than B, because A spent no time in free fall.

H General relativity alone cannot explain the final result.

I The result may be derived from the principle of equivalence.

Question 8 (*4 marks*) Select from the key for Q8 the **FOUR** items that should be included in a concise but complete account of the field equations of general relativity.

KEY for Q8

A energy density

B Hubble parameter

C momentum density

D momentum flux

E Ricci curvature

F Riemann curvature

G Robertson–Walker metric

H Schwarzschild metric

Question 9 (*4 marks*) Select from the key for Q9 the **FOUR** statements about black holes that are believed to be *true*.

KEY for Q9

A Every black hole gives rise to a spherically symmetric metric.

B A black hole can have an observable charge.

C The spacetime curvature at the event horizon of a black hole is infinite.

D The locally measured speed of light goes to zero at the event horizon.

E Nothing within the static limit can escape from a Kerr black hole.

F A black hole can act as a gravitational lens.

G A black hole's spacetime singularity may be modified by quantum effects.

H Black holes may radiate.

Question 10 (4 marks) The Robertson–Walker metric takes the form

$$(\Delta S)^2 = c^2(\Delta t)^2 - R^2(t)\left[\frac{(\Delta\sigma)^2}{1-k\sigma^2} + \sigma^2(\Delta\theta)^2 + \sigma^2\sin^2\theta(\Delta\phi)^2\right].$$

Select from the key for Q10 the **FOUR** statements that are *true*.

KEY for Q10

A This form of metric satisfies the cosmological principle.

B A galaxy cluster near $\sigma = 0$ occupies a privileged position in the universe.

C The metric would make no sense for $k > 0$, since $1 - k\sigma^2$ might then vanish.

D The evolution of $R(t)$ is determined solely by the pressure.

E If $k = 0$, $R(t)$ is constant.

F If $k < 0$, $R(t)$ cannot decrease with time.

G If $k = 0$, the angles of a large triangle sum to 180°.

H If $k < 0$, the angles of a large triangle sum to more than 180°.

I If $k = 0$, the circumference of a circle with constant σ is $2\pi R(t)\sigma$.

J If $k < 0$, the circumference of a circle with constant σ is less than $2\pi R(t)\sigma$.

Question 11 (*3 marks*) Suppose that the temperature of the cosmic background radiation is T_1 at time t_1 and is $T_2 = 0.99T_1$ at time t_2.

For each of (*a*) and (*b*), select the **ONE** answer from the key for Q11 that is closest to your answer.

(*a*) What is the ratio of the large-scale energy density of radiation at time t_2 to that at time t_1? ☐

(*b*) What is the ratio of the large-scale energy density of matter at time t_2 to that at time t_1? ☐

KEY for Q11

A 0.96

B 0.97

C 0.98

D 0.99

E 1.01

F 1.02

G 1.03

H 1.04

Question 12 (4 marks) Fill in the blanks with appropriate letters from the key, using none more than once.

The inflationary scenario seeks to solve a variety of cosmological puzzles, which include the following.

1) The problem of close-to-critical density. This arises because the present energy density is fairly close to the critical value determined by the ☐ . In the early universe, the ratio of these values must have been exceedingly close to unity, corresponding to a spatial curvature enormously smaller than the ☐ .

2) The problem of early homogeneity. This concerns the high degree of isotropy in the microwave background. At the time of decoupling, regions with significantly different ☐ could not have been in ☐ , according to previous cosmological models.

3) The problem of missing relics. Another perceived puzzle is that no-one has detected ☐ , which some physicists believe may have been copiously produced in the early universe.

A speculative solution is that quantum effects in the very early universe induced a ☐ in the field equations of general relativity. This would result in ☐ , which is called inflation. It is supposed that the inflationary epoch was followed by a process in which particles were created from vacuum energy, liberated in a manner akin to the release of ☐ .

KEY for Q12

A causal contact

B comoving coordinates

C cosmological constant

D exponential expansion

E Hubble parameter

F latent heat

G magnetic monopoles

H spacetime curvature

Question 14 The following 4 events are observed in an inertial frame S.

A spaceprobe, P, travelling in the positive x-direction at constant speed $c/2$, emits a light pulse L in the negative x-direction. This is event $\mathcal{E}_A$. It occurs at time $t = 0$ and at the origin of S.

P continues at its constant velocity until it is brought abruptly to rest by a collision. This is event $\mathcal{E}_B$. It occurs at time $t = 2T$ in S.

At the same time that P is brought to rest, the light pulse L is reflected by a mirror, set perpendicular to the x-axis. This is event $\mathcal{E}_C$.

P remains at rest and L travels in the positive x- direction until its reception by P, which is event $\mathcal{E}_D$. It occurs at time $t = 5T$ in S.

(a) Draw a clearly labelled spacetime diagram on which you show the four events, with their coordinates (ct, x), and the world-lines of P and L. *(4 marks)*

(b) Find the square of the invariant interval separating each of the following pairs of events: (i) $\mathcal{E}_A$ and $\mathcal{E}_B$; (ii) $\mathcal{E}_A$ and $\mathcal{E}_C$; (ii) $\mathcal{E}_B$ and $\mathcal{E}_D$. Your answers should be multiples of $(cT)^2$. *(3 marks)*

(c) Suppose that P was carrying a clock C that was set to zero at $\mathcal{E}_A$. Calculate its reading at $\mathcal{E}_D$, showing your working. *(2 marks)*

(d) On a second spacetime diagram, draw the world-line that C should have followed between $\mathcal{E}_A$ and $\mathcal{E}_D$ in order to *maximize* the difference between its initial and final readings. *Explain why this gives the maximum.* *(2 marks)*

(e) Calculate the percentage by which the maximum possible elapsed proper time between $\mathcal{E}_A$ and $\mathcal{E}_D$ exceeds the reading that you found in part (c). *(2 marks)*

Question 15 In answering this question, you may use the approximation $\sqrt{1-x} \approx 1 - x/2$, when $|x|$ is small compared to unity. You will need the metric outside an isolated, static, spherically symmetric body of mass M, which is

$$(\Delta\tau)^2 = \left(1 - \frac{k}{r}\right)(\Delta t)^2 - \frac{1}{c^2}\left\{\frac{(\Delta r)^2}{1 - \frac{k}{r}} + r^2(\Delta\theta)^2 + r^2\sin^2\theta\,(\Delta\phi)^2\right\}$$

where $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$, $c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$ and $k = 2MG/c^2$.

(a) One of two identical lasers is placed on the surface of a planet (of mass M and radius R, greater than k) to radiate radially outwards towards an astronaut who has the other laser on board his ship, which hovers above the planet at effectively an infinitely great distance, and constant values of θ and ϕ.

The astronaut compares the frequency, f, of light received from the planet-bound laser with f_0, the frequency of his on-board laser. Show that

$$f = f_0\sqrt{1 - \frac{k}{R}}.$$

Give your reasoning. (*4 marks*)

(b) If the planet's mass and radius are 3×10^{26} kg and 10^6 m respectively, show that k/R is much less than 1. Derive a numerical value for the fractional shift $(f - f_0)/f_0$. (*4 marks*)

(c) Instead of a planet, suppose the mass distribution were that of a static black hole, with an event horizon at $r = k$, and that the transmitting laser sits on a space platform located at $r = \frac{4}{3}k$. What now is $(f - f_0)/f_0$? (*3 marks*)

(d) Briefly describe what the distant astronaut would see if the space platform were allowed to fall freely in towards the event horizon. (*2 marks*)

Question 16 This question concerns a homogeneous, *matter-dominated* universe, for which Einstein's field equations lead to:

$$\frac{1}{R(t)}\frac{\mathrm{d}^2R(t)}{\mathrm{d}t^2} = -\frac{4\pi G}{3c^2}\rho(t)$$

$$\left(\frac{1}{R(t)}\frac{\mathrm{d}R(t)}{\mathrm{d}t}\right)^2 + \frac{kc^2}{R^2(t)} = \frac{8\pi G}{3c^2}\rho(t)$$

where $R(t)$ and $\rho(t)$ are the scale factor and energy density at universal time t, k is the spatial curvature parameter, and

$$\frac{3c^2}{8\pi G} = 1.6 \times 10^{26}\,\mathrm{kg\,m^{-1}}.$$

(*a*) Show that such a universe will be closed if the energy density at time t_0 exceeds the critical value

$$\rho_c(t_0) = \frac{3c^2}{8\pi G}H^2(t_0)$$

where $H(t_0)$ is the Hubble parameter at time t_0. (*2 marks*)

(*b*) Show that

$$\frac{\rho(t_0)}{\rho_c(t_0)} = 2q(t_0)$$

where $q(t_0)$ is the deceleration parameter at time t_0 defined by

$$q(t)H^2(t) = -\frac{1}{R(t)}\frac{\mathrm{d}^2R(t)}{\mathrm{d}t^2}.$$ (*2 marks*)

(*c*) Suppose that it is found at time t_0 that

$$H(t_0) = 10^{-18}\,\mathrm{s^{-1}} \qquad \text{and} \qquad \rho(t_0) = 3.2 \times 10^{-10}\,\mathrm{J\,m^{-3}}.$$

(*i*) Show that this universe is closed.

(*ii*) Draw a *rough sketch* that shows the evolution of the scale factor for times greater than t_0.

(*iii*) *Explain* why values of the deceleration parameter less than 0.5 can never be observed in this universe.

(*iv*) What can be said about the angles of a triangle formed by three light signals connecting three widely separated clusters of galaxies in this universe?

(*v*) If t_1 is the time at which the energy density has its minimum value, what is the value of $H(t_1)$?

(*vi*) What is the value of $q(t_1)$?

(*vii*) Given that $R(t_1) = 2R(t_0)$, what is the minimum energy density for this matter-dominated universe?

(*7 marks*)

(*d*) Suppose that at t_0 this universe contained background radiation whose contribution to the total energy density was merely 0.016%, consistent with the assumption of matter domination. What will be the percentage contribution of radiation at time t_1? (*2 marks*)

Question 17 Describe the meaning of the word *redshift* and the various ways in which it can arise in the context of this course. In particular describe and explain the characteristics the 3 K cosmic microwave background radiation, and comment on its significance for our understanding of the past and future of the universe. (*13 marks*)

Aim to use fewer than about 300 words. Marks will be given primarily for the relevance and clarity of your explanation.

Question 18 Describe the main experimental tests of the general theory of relativity. (*13 marks*)

Aim to use fewer than about 300 words. Marks will be given primarily for the relevance and clarity of your explanation.

[END OF QUESTION PAPER]