

It follows that

$$\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} H_2 = \left\{ \begin{pmatrix} 2 & x \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R}, y \neq 0 \right\}. \quad 1A$$

Similarly, the right coset is

$$\begin{aligned} H_2 \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} &= \left\{ \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} : b, d \in \mathbb{R}, d \neq 0 \right\} \\ &= \left\{ \begin{pmatrix} 2 & 3+4b \\ 0 & 4d \end{pmatrix} : b, d \in \mathbb{R}, d \neq 0 \right\}. \end{aligned} \quad 1M$$

Again, $4d$ varies over \mathbb{R}^* , and $3+4b$ varies over \mathbb{R} , so that

$$H_2 \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} = \left\{ \begin{pmatrix} 2 & x \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R}, y \neq 0 \right\}. \quad 1A$$

Question 19

- (a) The required affine transformation t is of the form

$$t(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}.$$

First, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

Next, the first column of the matrix \mathbf{A} is

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix},$$

and its second column is

$$\begin{pmatrix} 1 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \end{pmatrix}.$$

Thus $\mathbf{A} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}.$

It follows that the formula for t is

$$t(\mathbf{x}) = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad 2A$$

- (b) Now the inverse of the matrix $\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$ is $\frac{1}{36} \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/9 \end{pmatrix},$

and

$$\begin{pmatrix} 1/4 & 0 \\ 0 & 1/9 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}.$$

It follows that the inverse affine transformation is

$$t^{-1}(\mathbf{x}) = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/9 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}.$$

Hence the image of the point (x, y) is $(\frac{1}{4}(x-1), \frac{1}{9}y).$

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- (c) Let $(x', y') = t^{-1}(x, y)$; then $x' = \frac{1}{4}(x-1)$ and $y' = \frac{1}{9}y.$

It follows that the image of the ellipse under the affine transformation t^{-1} has equation

$$(x')^2 + (y')^2 = 1;$$

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that is, the unit circle.

- (d) Since affine transformations preserve tangency, under the mapping t^{-1} the tangents to the ellipse at P, Q map to tangents to the unit circle at $P', Q',$ say, that meet at $M',$ the image of M under $t.$

2M

