PART II

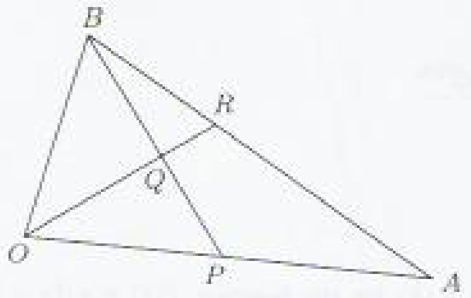
(i) *You should attempt* **no more than THREE** *questions from this part.*

(ii) *Each question carries 10 marks. The mark allocation for each section of a question is given in square brackets beside the section.*

(iii) *Start each question on a new page of your answer book.*

Question 15

The triangle OAB has vertices at O (the origin) and at points A and B with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The point P is the midpoint of OA and Q is the midpoint of PB. The line OQ meets AB at R.



(a) Write down the position vectors $\mathbf{p}$ and $\mathbf{q}$ of the points P and Q in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

(b) Find the vector form of the equations of the lines AB and OQ, and hence find the position vector of the point R. In what ratio does R divide AB? [5]

(c) If $\mathbf{a} = (4, -3)$ and $\mathbf{b} = (1, 3)$ show that OR is perpendicular to AB, and find the cosine of the angle $A\hat{O}R$. [3]

Question 16

The matrix $\mathbf{A} = \begin{pmatrix} 4 & 2 & -4 \\ 2 & 5 & 0 \\ -4 & 0 & 5 \end{pmatrix}$ represents the linear transformation $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with respect to the standard basis in both the domain and the codomain.

(a) Verify that $\mathbf{e}_1 = (5, -2, 4)$ and $\mathbf{e}_2 = (2, 1, -2)$ are eigenvectors of $\mathbf{A}$, and write down the corresponding eigenvalues. [2]

(b) Find a vector $\mathbf{e}_3$ such that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. Show that $\mathbf{e}_3$ is an eigenvector of $\mathbf{A}$, and write down the corresponding eigenvalue. [3]

(c) Express the vector $(1, 2, 2)$ in terms of your eigenvector basis. [2]

(d) Write down a diagonal matrix $\mathbf{D}$ and find a matrix $\mathbf{P}$ such that

$$\mathbf{D} = \mathbf{P}^T\mathbf{A}\mathbf{P}.$$ [3]

Question 17

(a) Determine whether or not the following functions are continuous at 0.

$$f(x) = \begin{cases} 1 + 2x^2 & x > 0; \\ \cos x & x \leq 0. \end{cases}$$

$$g(x) = \begin{cases} 0 & x = 0; \\ x^2 \sin\left(\dfrac{1}{x^3}\right) & x \neq 0. \end{cases}$$ [6]

(b) Show that the equation $x^2 + \sin(x/2) = 1$ has exactly one solution in $[0, \pi]$.

State clearly any results that you use. $2x + \frac{1}{2}\cos(x/2) = 0$ [4]

$\cos(x/2) = -4x$