(b) If $|\epsilon| << 1$ both a and θ change slowly and if $\omega >> \epsilon$, ϕ changes rapidly by comparison with θ. Thus when ϕ changes by 2π both a and θ change by a quantity of $O(\epsilon)$ and we can replace the right-hand sides of the equations for $\dot{a}$ and $\dot{\theta}$ by their averages over ϕ, assuming a and θ are constant:

$$\frac{da}{dt} = -\frac{\epsilon}{\omega}[A(a) + O(\epsilon)],$$
$$\frac{d\theta}{dt} = -\frac{\epsilon}{a\omega}[B(a) + O(\epsilon)],$$

where

$$A(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi \sin\phi\, f(a\cos\phi, -a\omega\sin\phi),$$
$$B(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi \cos\phi\, f(a\cos\phi, -a\omega\sin\phi).$$

(c) In this case, $f = -\dot{x}|\dot{x}| = a^2\omega^2 \sin\phi|\sin\phi|$, so $B(a) = 0$ and

$$A(a) = \frac{a^2\omega^2}{2\pi}\int_{-\pi}^{\pi} d\phi \sin^2\phi|\sin\phi| = \frac{4a^2\omega^2}{3\pi}$$

and the equations of motion, to $O(\epsilon)$, are

$$\frac{da}{dt} = -\frac{4\epsilon\omega a^2}{3\pi}, \quad \frac{d\theta}{dt} = 0.$$

These equations can be integrated to give

$$a = a_0\left[1 + \frac{4\epsilon\omega a_0}{3\pi}t\right]^{-1}, \quad \theta = \theta_0,$$

(a_0, θ_0) being the values of a and θ at time $t = 0$. Thus

$$x = \frac{a_0}{\left[1 + \frac{4\epsilon\omega a_0 t}{3\pi}\right]}\cos(\omega t + \theta_0) + O(\epsilon^2).$$

The term $O(\epsilon^2)$ which was ignored in the above derivation usually becomes comparable to the first term, of $O(\epsilon)$, when $\epsilon^2 t \sim \epsilon$, i.e. $t \sim 1/\epsilon$.

Question 4

(a) Since $\cos\omega t = T_n(y/A)$ and $\ddot{y} = -(\omega/n)^2 y$, substitution into the equation of motion gives

$$-\left(\frac{\omega}{n}\right)^2 y + g(y) = FT_n\left(\frac{y}{A}\right).$$

(b) Since $T_3(z) = 4z^3 - 3z$, the above equation for g becomes

$$g(y) = 4F(y/A)^3 + y[\omega^2/9 - 3F/A].$$

But g is given as $g = y + \epsilon y^3$ so equating coefficients of powers of y gives

$$4F/A^3 = \epsilon \quad \text{and} \quad \omega^2/9 - 3F/A = 1$$

or

$$A = (4F/\epsilon)^{1/3} \quad \text{and} \quad \omega^2 = 9\left[1 + 3(\epsilon F^2/4)^{1/3}\right].$$