Question 5

(a) With the given form for $x(t)$ we have

$$x^2 = c^2 + a^2/2 + b^2/2 + 2ac\cos\omega t + 2bc\sin\omega t + \text{higher harmonics}$$

and substituting into the equation of motion and equating coefficients gives the equations

$$c = -\epsilon\left[c^2 + \frac{a^2+b^2}{2}\right], \tag{1}$$

$$F = 2a[\beta + c], \tag{2}$$

$$0 = 2b[\beta + c]. \tag{3}$$

Equation (3) shows that either $b = 0$ or $\beta + c = 0$ and from (2), since $F \neq 0$, we must have $b = 0$. Then (1) shows that $c = O(\epsilon)$ so c may be ignored in (2) to give $a = F/2\beta$; then (1) gives $c = -\epsilon a^2/2$. Hence

$$x = -\epsilon F^2/8\beta^2 + (F/2\beta)\cos\omega t + O(\epsilon^2).$$

(b) Allowing a, b and c to be functions of $\tau = \epsilon t$ and denoting differentiation with respect to τ by a prime, we have

$$\ddot{x} = -a\omega^2\cos\omega t - b\omega^2\sin\omega t + 2\epsilon\omega[b'\cos\omega t - a'\sin\omega t] + O(\epsilon^2).$$

Substituting this into the equation of motion, using the above approximation to x^2 and equating coefficients gives

$$c = -\epsilon\left[c^2 + \frac{a^2+b^2}{2}\right],$$

$$2\epsilon\omega a' = b(1-\omega^2) + 2bc\epsilon,$$

$$2\epsilon\omega b' = -a(1-\omega^2) - 2ac\epsilon + \epsilon F.$$

Since $c = O(\epsilon)$ and $\omega^2 = 1 - 2\epsilon\beta$, these give

$$a' = \beta b + O(\epsilon),$$

$$b' = -a\beta + F/2 + O(\epsilon),$$

$$c = -\epsilon(a^2+b^2)/2,$$

with general solution

$$a(t) = F/2\beta + A\sin(\beta\epsilon t + \alpha),$$

$$b(t) = A\cos(\beta\epsilon t + \alpha),$$

for some constants A and α. Since a, b and c are bounded, the periodic solution found above is stable, at least for the time during which this approximation is valid, $t \sim 1/\epsilon$.

Question 6

Put $\theta = \omega t$ and denote derivatives with respect to θ by primes. Then on using the expansions for ω and $x(\theta)$ and ignoring terms $O(\epsilon^3)$,

$$(1 + \epsilon\omega_1 + \epsilon^2\omega_2)[x_0'' + \epsilon x_1'' + \epsilon^2 x_2'' + \epsilon(x_0' + \epsilon x_1')^2] + x_0 + \epsilon x_1 + \epsilon^2 x_2 = 0.$$

Expanding this and equating each power of ϵ to zero gives

$$x_0'' + x_0 = 0, \tag{1}$$

$$x_1'' + x_1 = -x_0'^2 - 2\omega_1 x_0'', \tag{2}$$

$$x_2'' + x_2 = -2\omega_1(x_1'' + x_0'^2) - (\omega_1^2 + 2\omega_2)x_0'' - 2x_0'x_1'. \tag{3}$$

In solving these we need a periodic solution with the initial conditions given. This can be achieved if $x_0(0) = [illegible]$, $x_k(0) = 0, k = 1, 2, \ldots,$ and $x_k'(0) = 0$, $k = 0, 1, \ldots,$ and if all $x_k(\theta)$ are periodic.

Now the R.H.S. of Equation [illegible] contains $x_0, x_1, \ldots, x_n$ and $\omega_0 = 1, \omega_1, \ldots, \omega_n$ only. In each case [illegible] determined by the condition that each x_n is 2π-periodic.