(c) Putting $x = 1 + z$ gives

$$\frac{d^2z}{dt^2} - 1 - z + (1 + 4z + 6z^2 + 4z^3 + z^4) = 0$$

or

$$\frac{d^2z}{dt^2} + 3z + 6z^2 + O(z^3) = 0.$$

Near $z = 0$, $V(z) = 3z^2/2 + 2z^3$ and clearly $V'' < 0$ at $z = 0$, so the fixed point at $x = 1$ is a centre.

Now apply the method of harmonic balance to this last equation. As a first attempt we might try

$$z = a\cos\omega t,$$

but as

$$\begin{aligned} z^2 &= a^2\cos^2\omega t = a^2(1 + \cos 2\omega t)/2 \\ &= a^2/2 + \text{higher harmonics} \end{aligned}$$

we would get $a = 0$ on equating the constant term and the coefficient of the first harmonic to zero. The reason for this is that the mean value of z^2 over the small amplitude motion (obtained by ignoring the $6z^2$ term) is non-zero so that the $6z^2$ term has the effect of moving the 'centre of gravity' of motion away from $z = 0$. Thus the correct trial function is

$$z = b + a\cos\omega t,$$

where b is small. Then

$$z^2 = b^2 + a^2/2 + 2ab\cos\omega t + \text{higher harmonics}$$

and the substitution into the equation of motion gives

$$-a\omega^2\cos\omega t + 3b + 3a\cos\omega t + 6b^2 + 3a^2 + 12ab\cos\omega t = 0.$$

Thus we have

$$\begin{aligned} &b + a^2 + 2b^2 = 0, \\ &3a - a\omega^2 + 12ab = 0. \end{aligned}$$

Since b is small the first equation gives $b = -a^2$. Then the second gives $\omega^2 = 3(1 - 4a^2)$. The constant a must be small otherwise the z^2 term in the equation of motion becomes important and then the ignored higher harmonics are also important.

Question 2

(a) The fixed points of the system are those points at which $\dot{x} = \dot{y} = 0$ for all t, i.e. at the roots of

$$X(x,y) = 0, \qquad Y(x,y) = 0.$$

If (a, b) is a fixed point and is a simple zero of the above equations, the linear approximation to the equations of motion in the neighbourhood of (a, b) is

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix}, \qquad x = a + u, \quad y = b + v,$$

where the non-singular 2×2 matrix A is

$$A = \begin{pmatrix} \dfrac{\partial X}{\partial x} & \dfrac{\partial X}{\partial y} \\ \dfrac{\partial Y}{\partial x} & \dfrac{\partial Y}{\partial y} \end{pmatrix}$$

all derivatives being evaluated at (a, b). The fixed points can be categorised according to the values of the eigenvalues of A and these, in turn, depend on the two quantities $\det(A)$ and $\mathrm{tr}(A)$ only.