M821 Solutions to the Specimen Examination Paper

Question 1

(a) (i) A system is autonomous if neither X nor Y depends explicitly upon the time t:

$$\frac{\partial X}{\partial t} = \frac{\partial Y}{\partial t} = 0, \qquad \text{all} x, y, t.$$

(ii) A system is conservative if there exists a continuously differentiable function $f(x, y, t)$ which is constant for any solution $(x(t), y(t))$ of the system: the value of the function f is conserved along a trajectory.

Along a trajectory we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial t}.$$

Therefore for f to be constant

$$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t} = 0, \quad \text{along } (x(t), y(t)).$$

This is the general definition: the only cases dealt with in **JS** are those for which

$$X(x, y, t) = y, \qquad Y(x, y, t) = -\frac{dV}{dx}(x),$$

with

$$f = \tfrac{1}{2}y^2 + V(x).$$

Full marks would be given for this partial solution.

(b) Putting $\dot{x} = y$ gives

$$\dot{x} = y, \quad \dot{y} = -V'(x),$$

so the fixed points are at $y = 0$ and the roots of $V'(x) = 0$. Putting $x = x_i + z$ and expanding gives

$$V'(x) = V'(x_i) + zV''(x_i) + O(z^2),$$

with $V''(x_i) \neq 0$ as the roots are simple. The equations of motion near the fixed point are then

$$\begin{pmatrix} \dot{z} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -V''(x_i) & 0 \end{pmatrix} \begin{pmatrix} z \\ y \end{pmatrix}.$$

The eigenvalues of the matrix are

$$\lambda^2 = -V''(x_i).$$

Thus if $V''(x_i) < 0$ the roots are real and of different sign so the fixed point is a saddle. If $V''(x_i) > 0$ the roots are purely imaginary, giving a centre.

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