Question 3

(a) For the system

$$\frac{d^2x}{dt^2} + \omega^2 x = \epsilon f(x, \frac{dx}{dt})$$

show that if

$$x = a(t)\cos(\omega t + \theta(t)),$$

where a and θ are chosen so that

$$\frac{dx}{dt} = -\omega\, a(t)\sin(\omega t + \theta(t)), \qquad \text{for all } t,$$

then $a(t), \theta(t)$ satisfy the equations

$$\frac{da}{dt} = -\frac{\epsilon}{\omega}\sin\phi\ f(a\cos\phi, -a\omega\sin\phi),$$
$$\frac{d\theta}{dt} = -\frac{\epsilon}{a\omega}\cos\phi\ f(a\cos\phi, -a\omega\sin\phi),$$

where $\phi = \omega t + \theta(t)$.

(b) Using these equations derive the approximate equations

$$\frac{da}{dt} = -\frac{\epsilon}{\omega}A(a)$$
$$\frac{d\theta}{dt} = -\frac{\epsilon}{a\omega}B(a),$$

stating carefully all assumptions made and giving explicit expressions for the functions $A(a)$ and $B(a)$.

(c) Hence, or otherwise, find an approximate solution to the equation

$$\frac{d^2x}{dt^2} + \omega^2 x = -\epsilon\frac{dx}{dt}\left|\frac{dx}{dt}\right| \qquad |\epsilon| << 1$$

and state, without proof, the time interval for which the approximation is within $0(\epsilon)$ of the exact solution.

Question 4

(a) The Chebyshev polynomial of order n, T_n, is defined by the relation

$$T_n(\cos x) = \cos nx, \qquad n = 0, 1, 2, \ldots,$$

so that $T_0(z) = 1$, $T_1(z) = z$, $T_2(z) = 2z^2 - 1$ etc.

Use this definition to show that the equation

$$\frac{d^2y}{dt^2} + g(y) = F\cos\omega t,$$

F being a constant, has a subharmonic solution of order n given by

$$y = A\cos\left(\frac{\omega t}{n}\right)$$

if

$$g(y) = \left(\frac{\omega}{n}\right)^2 y + FT_n\left(\frac{y}{A}\right).$$

(b) Hence, or otherwise, show that the equation

$$\frac{d^2y}{dt^2} + y + \epsilon y^3 = F\cos\omega t$$

has a subharmonic solution of order 3 if A and ω satisfy

$$A = (4F/\epsilon)^{1/3}, \quad \omega^2 = 9\left[1 + 3(\epsilon F^2/4)^{1/3}\right].$$

You may use the result

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$