In the particular case given, $x_0 = A\cos\theta$ so (2) becomes

$$x_1'' + x_1 = -A^2\sin^2\theta + 2\omega_1 A\cos\theta$$
$$= -A^2(1-\cos 2\theta)/2 + 2\omega_1 A\cos\theta.$$

For x_1 to be periodic we must have $\omega_1 = 0$, then

$$x_1 = B\sin\theta + C\cos\theta - A^2/2 - (A^2/6)\cos 2\theta,$$

and the initial conditions give $C = 2A^2/3, B = 0$. Equation (3) becomes

$$x_2'' + x_2 = 2\omega_2\ A\cos\theta + 2A^3\sin\theta(\sin 2\theta - 2\sin\theta)/3.$$

Using the relations

$$\sin^2\theta = (1-\cos 2\theta)/2 \quad \text{and} \quad \sin 2\theta\sin\theta = (\cos\theta - \cos 3\theta)/2$$

gives

$$x_2'' + x_2 = -2A^3/3 + (2\omega_2 A + A^3/3)\cos\theta + (2A^3/3)\cos 2\theta - (A^3/3)\cos 3\theta.$$

The only way in which x_2 can be periodic is for the coefficient of $\cos\theta$ to be zero and hence $\omega_2 = -A^2/6$. Therefore

$$\omega = 1 - \epsilon^2 A^2/6 + O(\epsilon^3)$$

and

$$x = A\cos\theta - \epsilon A^2(3 - 4\cos\theta + \cos 2\theta)/6, \quad \theta = \omega t.$$

Question 7

(a) The characteristic numbers are the eigenvalues of the matrix E where $\Phi(t+2\pi) = \Phi(t)E$. But since $\Phi(0) = I, E = \Phi(2\pi)$. The eigenvalues are given by

$$\begin{vmatrix} \Phi_{11} - \mu & \Phi_{12} \\ \Phi_{21} & \Phi_{22} - \mu \end{vmatrix} = 0$$

or

$$\mu^2 - \mu\,\mathrm{tr}(\Phi(2\pi)) + \det(\Phi(2\pi)) = 0.$$

(b) The A-matrix for the second-order system is

$$A(t) = \begin{pmatrix} 0 & 1 \\ -f(t) & 0 \end{pmatrix},$$

so $\mathrm{tr}A = 0$ and $\det(\Phi(2\pi)) = \det(\Phi(0)) = 1$, hence the result.

(c) In order to apply this result we need explicit expressions for Φ_{11} and Φ_{22}. No $\Phi_{11}(t)$ is the solution with $\Phi_{11}(0) = 1$ and $\Phi_{21}(0) = 0$; so

$$\Phi_{11}(t) = \begin{cases} \cos\omega_1 t, & 0 \le t < \pi \\ A\cos\omega_2(t-\pi) + B\sin\omega_2(t-\pi), & \pi \le t < 2\pi, \end{cases}$$

where the coefficients A and B are obtained by matching the two soluti[illegible] at $t = \pi$, giving $A = \cos\omega_1\pi$, $B = -(\omega_1/\omega_2)\sin\omega_1\pi$ and

$$\Phi_{11}(2\pi) = \cos\omega_1\pi\cos\omega_2\pi - \frac{\omega_1}{\omega_2}\sin\omega_1\pi\sin\omega_2\pi.$$

Similarly, the $\Phi_{12}(t)$ solution satisfies $\Phi_{12}(0) = 0, \Phi_{22}(0) = \dot\Phi_{12}(0) =$ [illegible] so

$$\Phi_{12} = \begin{cases} \omega_1^{-1}\sin\omega_1 t, & 0 \le t < \pi, \\ C\cos\omega_2(t-\pi) + B\sin\omega_2(t-\pi), & \pi \le t < 2\pi \end{cases}$$

as above, $C = \sin\omega_1\pi/\omega_1$, $B = \cos\omega_1\pi/\omega_2$ and

$$\Phi_{22}(2\pi) = \dot\Phi_{12}(2\pi) = \cos\omega_1\pi\cos\omega_2\pi - \frac{\omega_2}{\omega_1}\sin\omega_1\pi\sin\omega_2\pi$$

Thu[illegible]

$$\mathrm{tr}(\Phi(2\pi)) = 2\left(\cos\omega_1\pi\cos\omega_2\pi - [illegible]\sin\omega_1\pi\sin\omega_2\pi\right) = 2T.$$

The characteristic numb[illegible]s are then

$$\mu = T \pm (T^2 - 1)^{\frac{1}{2}}.$$

For stability we require $\mu \le 1$, that is [illegible] the [illegible] condition.