Question 1

(a) What does it mean for the system

$$\frac{dx}{dt} = X(x, y, t),\ \frac{dy}{dt} = Y(x, y, t)$$

to be

(i) autonomous, and

(ii) conservative?

(b) For the system described by a second-order equation of the form

$$\frac{d^2x}{dt^2} = -\frac{dV}{dx},$$

where $V(x)$ is a differentiable function of x only, show that the equilibrium (or fixed) points in the phase plane, $(x, \dot{x})$, are at $(x_i, 0)$, where x_i are the roots of

$$\frac{dV}{dx} = 0.$$

Assuming that this equation has only simple roots show that fixed points are either saddles or centres: find the condition on V at x_i that the fixed point at $(x_i, 0)$ should be a centre.

(c) Show that the system

$$\frac{d^2x}{dt^2} - x + x^4 = 0$$

has a centre at $(1, 0)$ and that in the neighbourhood of this centre the equation can be approximated by

$$\frac{d^2z}{dt^2} + 3z + 6z^2 = 0$$

where $x = 1 + z$.

Use the method of Harmonic Balance to show that an approximate solution is

$$x = 1 - a^2 + a\cos\omega t$$

where

$$\omega^2 = 3(1 - 4a^2)$$

and where a is a small constant.

Question 2

(a) For the two-dimensional system

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y)$$

define the term equilibrium (or fixed) point. On which properties of the functions X, Y does the linear categorization of fixed points depend?

(b) A particular molecular mechanism can be modelled by the equations

$$\frac{dx}{dt} = \frac{a}{A + ky} - b, \quad \frac{dy}{dt} = \alpha x - \beta, \quad (x, y > 0, a > bA)$$

where the constants a, A, k, b, α and β are all positive. Show that the system has only one fixed point and that this is a centre. Hence deduce that for small departures from the fixed point both x and y oscillate.

(c) Show that the phase curves of this model can be written in the form

$$f(y) = g(x) + \text{ constant.}$$

Find explicit forms for f and g and by sketching their graphs show that x and y always exhibit periodic oscillations.