Question 5

(a) Obtain an approximate solution of period $2\pi/\omega$ for the equation

$$\frac{d^2x}{dt^2} + x + \epsilon x^2 = \epsilon F \cos\omega t, \quad |\epsilon| << 1,$$

by assuming the form

$$x(t) = c + a\cos\omega t + b\sin\omega t$$

and where $\omega^2 = 1 - 2\epsilon\beta$ with $\beta = 0(1)$.

(b) Allow a, b, and c to be functions of $\tau = \epsilon t$ and show that, to $0(\epsilon)$, they satisfy the equations.

$$\frac{da}{d\tau} = \beta b,$$
$$\frac{db}{d\tau} = -\beta a + F/2,$$
$$c = -\epsilon(a^2 + b^2)/2.$$

Find explicit expressions for $a(\tau)$ and $b(\tau)$ and deduce that the periodic solution found in part (a) is stable.

Question 6

(a) For the equation

$$\frac{d^2x}{dt^2} + x + \epsilon\left(\frac{dx}{dt}\right)^2 = 0, \qquad |\epsilon| << 1,$$

with the intitial conditions $x(0) = A$, $\dot{x}(0) = 0$ use Lindstedt's method to show that the periodic solution with frequency

$$\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + 0(\epsilon^3)$$

is

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + 0(\epsilon^3)$$

where the x_i satisfy the equations,

$$x_0'' + x_0 = 0, \qquad x_0(0) = A, x_0'(0) = 0,$$
$$x_1'' + x_1 = -2\omega_1 x_0'' - x_0'^2,$$
$$x_2'' + x_2 = -2\omega_1(x_1'' + x_0'^2) - (\omega_1^2 + 2\omega_2)x_0'' - 2x_0'x_1',$$

with $x_k(0) = x_k'(0) = 0, k = 1, 2, \ldots$. State briefly how these equations should be solved to obtain the desired solution.

Show that

$$\omega = 1 - \epsilon^2 A^2/6 + 0(\epsilon^3)$$

and

$$x(t) = A\cos\theta - \epsilon A^2(3 - 4\cos\theta + \cos 2\theta)/6 + 0(\epsilon^2)$$

with $\theta = \omega t$.