Question 7

(a) If $\Phi(t) = (\phi_{ij}(t))$ is a fundamental matrix of the system

$$\dot{\mathbf{x}} = A(t)\mathbf{x},$$

$A(t)$ being a 2π-periodic 2×2 matrix, with $\Phi(0) = I$, show that the characteristic numbers, μ, satisfy the equation

$$\mu^2 - \mu \operatorname{tr}(\Phi(2\pi)) + \det(\Phi(2\pi)) = 0.$$

(b) Deduce that the characteristic numbers for the equation

$$\ddot{x} + f(t)x = 0,$$

$f(t)$ being a 2π-periodic function, satisfy

$$\mu^2 - \mu[\phi_{11}(2\pi) + \phi_{22}(2\pi)] + 1 = 0.$$

You may assume that

$$\det \Phi(t) = \det \Phi(t_0) \exp \int_{t_0}^{t} ds \ \operatorname{tr}(A(s)).$$

(c) Hence, or otherwise, show that when

$$f(t) = \begin{cases} \omega_1^2 & 0 \leqq t < \pi \\ \omega_2^2 & \pi \leqq t < 2\pi \end{cases}$$

$$f(t + 2\pi) = f(t)$$

the system is stable if

$$\left| \cos \omega_1 \pi \cos \omega_2 \pi - \frac{(\omega_1^2 + \omega_2^2)}{2\omega_1 \omega_2} \sin \omega_1 \pi \sin \omega_2 \pi \right| \leqq 1.$$

Question 8

Define Poincaré and Liapunov stability and explain their differences.

If $V(x)$ is even, differentiable function of x and monotonic increasing on $x > 0$ discuss the nature of the solutions of the equation

$$\frac{d^2 x}{dt^2} = -\frac{dV}{dx}.$$

In particular state which, if any, are Poincaré stable and which, if any, are Liapunov stable. Give an example of a function $V(x)$ for which all orbits are Liapunov stable.

[END OF QUESTION PAPER]