(b) $\dot{y} = 0$ when $x = \beta/\alpha > 0$ and this gives $y = (a - bA)/kb > 0$. Thus there is only one fixed point. The matrix A at this point is

$$A = \begin{pmatrix} 0 & -kb^2/a \\ \alpha & 0 \end{pmatrix}$$

with eigenvalues $\lambda = \pm i(\alpha k b^2/a)^{\frac{1}{2}}$. Thus the fixed point is a centre surrounded by closed orbits, that is x and y oscillate for small departures from the fixed point.

(c) The phase curves are solutions of the differential equation

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\alpha x - \beta}{\left[\frac{a}{A+ky} - b\right]},$$

which can be written in the form

$$\int dy \left(\frac{a}{A+ky} - b\right) = \int dx\,(\alpha x - \beta)$$

or

$$f(y) = g(x) + \text{constant},$$

where

$$f(y) = (a/k)\ln(A + ky) - by, \quad g(x) = \tfrac{1}{2}\alpha x^2 - \beta x.$$

The graphs of these functions are sketched below.

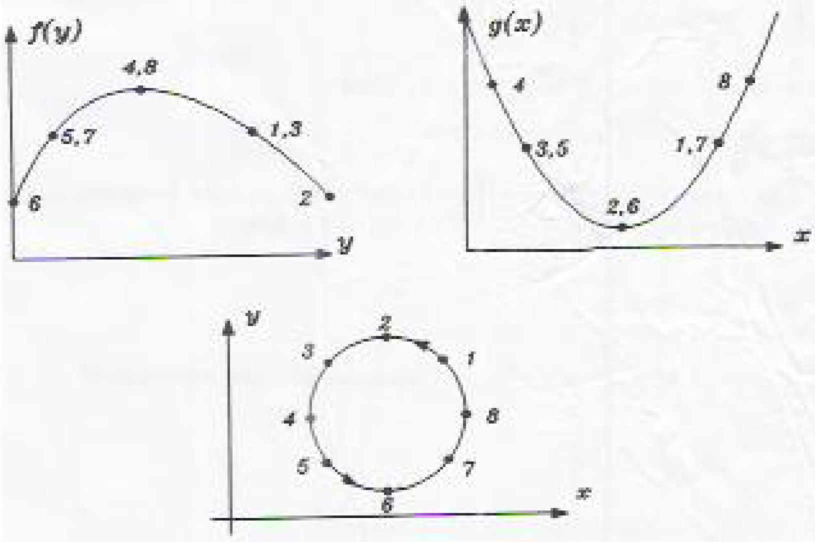

Figure 1

The fixed point found above is at the minimum of $g(x)$ and the maximum of $f(y)$. Imposing the condition $f - g =$ constant, and following the motion along the above graphs, shows that the phase curves are all closed and hence that the motion is periodic.

Question 3

(a) If $x = a\cos(\omega t + \theta)$ then

$$\dot{x} = -\omega a \sin(\omega t + \theta) + [\dot{a}\cos(\omega t + \theta) - a\dot{\theta}\sin(\omega t + \theta)].$$

The term in square brackets is zero for all t so

$$\ddot{x} = -\omega \dot{a} \sin\phi - \omega a(\omega + \dot{\theta})\cos\phi$$

and substitution into the equation of motion gives

$$-\omega\dot{a}\sin\phi - a\omega\dot{\theta}\cos\phi = \epsilon f(a\cos\phi, -a\omega\sin\phi).$$

But also $\dot{a}\cos\phi = a\dot{\theta}\sin\phi$, and these equations can be solved for $(\dot{a}, \dot{\theta})$ to give the required solutions.