Question 8

An orbit $\mathbf{x}(t), t \geq t_0$, for an autonomous system is Poincaré stable if for every $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that if

$$|\mathbf{x}(t_0) - \mathbf{y}(t_0)| < \delta$$

then the maximum distance between the two half-phase curves C and D, where

$$C = \{\mathbf{x}(t), t \geq t_0\}, \qquad D = \{\mathbf{y}(t), t \geq t_0\},$$

is less than ϵ.

On the other hand, for any system, autonomous or not, the orbit $\mathbf{x}(t), t \geq t_0$, is Liapunov stable if for any $\epsilon > 0$ there exists a $\delta(\epsilon, t_0)$ such that if

$$|\mathbf{x}(t_0) - \mathbf{y}(t_0)| < \delta$$

then

$$|\mathbf{x}(t) - \mathbf{y}(t)| < \epsilon \quad \text{for all } t \geq t_0.$$

The first obvious difference is that Poincaré stability applied only to autonomous systems. But for autonomous systems Poincaré stability is much less restrictive as it just says that the phase *curves* remain close, whereas Liapunov stability means that two phase *points* which are initially close remain close for *all* time; typically orbits near an asymptotically stable fixed point are Liapunov stable. Also, all Liapunov stable orbits are Poincaré stable, but not vice versa.

For the equation given we can choose, without loss of generality, $V(0) = 0$; then the function $V(x)$ and consequence phase curves look like those shown.

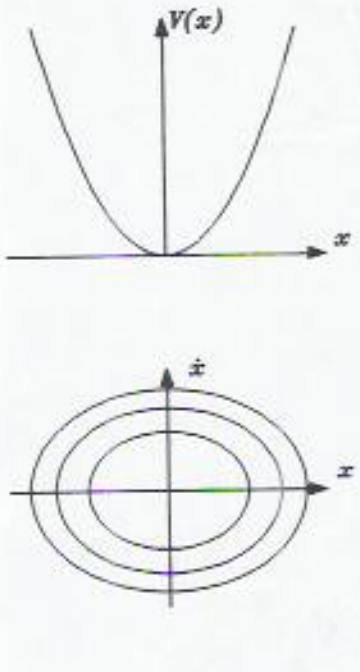

Figure 2

The centre at $(x, \dot{x}) = (0, 0)$ is clearly Liapunov stable. Further, all other phase curves are Poincaré stable.

All orbits are periodic, and in general the periods of an orbit and its neighbouring orbits are different. Therefore, apart from the origin, no orbits are Liapunov stable. But, if the periods of *all* orbits were the same then *all* orbits would be Liapunov stable. The only case when this happens is when $V(x)$ is quadratic:

$$V(x) = \tfrac{1}{2}\omega^2 x^2,$$

where ω is any real number. In this case the period of all orbits is $2\pi/\omega$. Note, no proofs of these statements are given as the question did not ask for any.