Answer as many questions as you wish. Full marks may be obtained by **complete** *answers to* **NINE** *questions, provided that no more than* **SIX** *questions have been selected from any one part. All questions carry equal marks.*

PART I METRIC AND TOPOLOGICAL SPACES

Question 1

(i) Let $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ be the function given by

$$f(x_1, x_2) = 1 + |x_1| + |x_2|, \quad (x_1, x_2) \in \mathbf{R}^2.$$

(a) Sketch the subset

$$f^{-1}([-1, 2])$$

of $\mathbf{R}^2$. [3]

(b) Write down $f(f^{-1}([-1, 2]))$. [2]

(ii) Let $g : A \longrightarrow B$ be a function, where A and B are non-empty sets.

(a) Prove that if $K \subset B$ then

$$g(g^{-1}(K)) \subseteq K.$$ [3]

(b) Prove that if $K \subset B$ and g is onto then

$$g(g^{-1}(K)) = K.$$ [3]

Question 2

Let $d : \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ be defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ |x + 1| + |y + 1| + |x - y|, & \text{if } x \neq y. \end{cases}$$

(i) Show that d is a metric on $\mathbf{R}$. [5]

(ii) Determine the following open balls with respect to the metric d:

(a) $B_1(0)$;

(b) $B_1(-1)$. [3]

(iii) Show that the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$f(x) = 1 + x, \quad x \in \mathbf{R},$$

is not (d, d)-continuous at -1. [3]

Question 3

Let $A = \{0, 1, 2, \dots,\}$ be the set of non-negative integers and let $\mathcal{V}$ be the set of subsets of A defined as follows: $V \in \mathcal{V}$ if $0 \notin V$ or $B \subset V$ where B is the set of odd positive integers.

(i) Show that $\mathcal{V}$ is a topology on A. [6]

(ii) Determine the topology induced by $\mathcal{V}$ on the set $\{0, 1, 2, 3\}$. [5]