Question 7

(i) Prove that if A_1 and A_2 are connected subspaces of a topological space such that $A_1 \cap A_2 \neq \varnothing$, then $A_1 \cup A_2$ is connected. [5]

(ii) For each positive integer n, let J_n be the line segment in $\mathbf{R}^2$ joining $(0,0)$ to $\left(\frac{1}{n}, 1\right)$ and let

$$K = \{(x, y) \in \mathbf{R}^2 : x \leq 0, y > 0\}.$$

Prove that the subspace

$$\left(\bigcup_{n=1}^{\infty} J_n\right) \cup K$$

of $\mathbf{R}^2$ with the usual topology is connected. [6]

Question 8

Let d be the metric on the set $\mathbf{N}$ of positive integers defined by

$$d(i, j) = \left|\frac{i}{i+1} - \frac{j}{j+1}\right|, \quad i, j \in \mathbf{N}.$$

[You need **not** check that d is a metric on $\mathbf{N}$.]

Consider the sequence (x_n) where $x_n = n$ for each positive integer n.

(i) Prove that (x_n) is a Cauchy sequence in the metric space $\{\mathbf{N}, d\}$. [4]

(ii) Show that if $k \in \mathbf{N}$ then

$$d(k, x_n) \geq \frac{1}{2k+2}$$

for each integer n such that $n \geq 2k + 1$. [3]

(iii) Deduce that the metric space $\{\mathbf{N}, d\}$ is not complete. [4]