Question 13

Categorize each of the following conics (which you may assume are non-degenerate) as one of the following:

(a) homeomorphic to the z-sphere;

(b) a double cover of the z-sphere branched over two points;

(c) a double cover of the z-sphere branched over two points and having a pinch point.

In each case locate the branch points and pinch point when they exist.

(i) $6z^2 + 4zw + w^2 + 4z + 4 = 0$; [6]

(ii) $z^2 + 6zw + 9w^2 + z - 4w + 1 = 0$. [5]

Question 14

(i) Write down the value of j for a given value of k such that there is a 2-fold covering by jT^2 of kT^2 branched over 4 points. [3]

(ii) Show pictorially that there is a branched 2-fold covering by $3T^2$ of T^2 branched over 4 points. [2]

(iii) Write down the value of j for a given value of k such that there is a 3-fold covering by jT^2 of kT^2 with total branch point order $\delta = 4$. [2]

(iv) Show pictorially that there are the following branched 3-fold coverings:

  (a) by $3T^2$ of T^2 branched over 2 points;

  (b) by $6T^2$ of $2T^2$ branched over 2 points;

  (c) by $3kT^2$ of kT^2 branched over 2 points. [4]

Question 15

Throughout this question, a denotes a positive real number for which $5 < a < 45$ and F is the function defined by

$$F(z) = z^4 + 4z^2 + a.$$

(i) Show that all the roots of $F(z)$ lie inside the circle $|z| = 3$. [$1\frac{1}{2}$]

(ii) Show that all the roots of $F(z)$ lie outside the circle $|z| = 1$. [$1\frac{1}{2}$]

(iii) Deduce that all the roots of $F(z)$ lie inside the annulus $A = \{z : 1 \le |z| \le 3\}$. [1]

You may now assume that a is also such that $F(z)$ has no real or purely imaginary zeros.

(iv) Show that $F(it)$ is purely real. Deduce that $V_{C_1}(F(z))$ is zero, where C_1 is the segment $\{it : 0 \le t \le 3\}$ of the imaginary axis. [2]

(v) Let $C_2 = \{x : 0 \le x \le 3\}$ and let γ denote the contour $\{z = 3e^{i\theta} : 0 \le \theta \le \pi/2\}$. Show that there is exactly one zero of $F(z)$ that lies in the region bounded by C_2, γ and C_1. [4]

(vi) Deduce the number of roots of $F(z)$ lying in each quadrant. [1]